Review exercise 1

1 $a = \frac{dv}{dt} = e^{2t}$

$v = \int e^{2t}\,dt = \frac{1}{2}e^{2t} + A$

When $t = 0, v = 0$

$0 = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2}$

Hence $v = \frac{1}{2}(e^{2t} - 1)$, as required.

> To find the acceleration, integrate the velocity with respect to time. Remember to include a constant of integration.

2 a $a = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = -3e^{-\frac{1}{6}t} + A$

When $t = 0, v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

> Using $\int e^{kt}\,dt = \frac{1}{k}e^{kt}A$, then
>
> $\int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = \frac{1}{2\times\left(-\frac{1}{6}\right)}e^{-\frac{1}{6}t} + A = -\frac{1}{\frac{1}{3}}e^{-\frac{1}{6}t} + A$
>
> $= -3e^{-\frac{1}{6}t} + A$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180...$

The speed of P when $t = 3$ is 11.2 m s^{-1} (3 s.f.)

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

> As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if $t = 120$, then $e^{-\frac{1}{6}t} \approx 2.06 \times 10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

3 a $a = \frac{dv}{dt} = 2\sin\frac{1}{2}t$

$v = \int 2\sin\frac{1}{2}t\ dt = -4\cos\frac{1}{2}t + A$

When $t = 0, v = 4$

$4 = -4 + A \Rightarrow A = 8$

Hence $v = 8 - 4\cos\frac{1}{2}t$

> Using the formula
>
> $\int \sin at\,dt = -\frac{1}{a}\cos at + A$,
>
> $\int 2\sin\frac{1}{2}t\,dt = -\frac{2}{\frac{1}{2}}\cos\frac{1}{2}t + A = -4\cos\frac{1}{2}t + A$

3 b The distance, s metres, travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is given by

$$s = \int_0^{\frac{\pi}{2}} \left(8 - 4\cos\frac{1}{2}t\right)\mathrm{d}t$$

$$= \left[8t - 8\sin\frac{1}{2}t\right]_0^{\frac{\pi}{2}}$$

$$= 4\pi - 8\sin\frac{\pi}{4} = 4\pi - \frac{8}{\sqrt{2}}$$

$$= 4\pi - 4\sqrt{2} = 4(\pi - \sqrt{2})$$

The change in the displacement of P between any two times, say t_1 and t_2, can be found by calculating the definite integral of the velocity between the limits t_1 and t_2. If P has not turned around, this will also give the distance travelled by P.
The particle in this question does turn around when $\frac{\mathrm{d}v}{\mathrm{d}t} = a = 0$ but that does not happen until $t = 2\pi$, so P does not turn round in the interval $0 \leqslant t \leqslant \frac{\pi}{2}$.

The distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is $4(\pi - \sqrt{2})$ m.

4 a $$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{3}{\sqrt{(t+4)}} = -3(t-4)^{-\frac{1}{2}}$$

$$v = -3\int (t+4)^{-\frac{1}{2}}\mathrm{d}t = \frac{-3(t+4)^{\frac{1}{2}}}{\frac{1}{2}} + A = A - 6(t+4)^{\frac{1}{2}}$$

As the acceleration is towards O, $\frac{\mathrm{d}v}{\mathrm{d}t}$, which is always measured in the direction of x increasing, is negative.

When $t = 0, v = 18$

$18 = A - 6 \times 2 \Rightarrow A = 30$

Hence

$v = 30 - 6(t+4)^{\frac{1}{2}}$

The velocity of P is $\left[30 - 6\sqrt{(t+4)}\right]$ m s^{-1}, as required.

b

$0 = 30 - 6(t+4)^{\frac{1}{2}}$

$(t+4)^{\frac{1}{2}} = 5 \Rightarrow t + 4 = 25 \Rightarrow t = 21$

$v = \frac{\mathrm{d}x}{\mathrm{d}t} = 30 - 6(t+4)^{\frac{1}{2}}$

There are three steps needed to solve part **b**. First you must find the value of t for which P is instantaneously at rest; that is when $v = 0$. You must also find x in terms of t by integrating the expression you proved in part **a**. Finally you substitute your value of t into your expression for x. It is a characteristic of harder questions at this level that you often have to construct for yourself the steps needed to solve a problem.

$$x = \int \left(30 - 6(t+4)^{\frac{1}{2}}\right)\mathrm{d}t = 30t - \frac{6(t+4)^{\frac{3}{2}}}{\frac{3}{2}} + B$$

$$= 30t - 4(t+4)^{\frac{3}{2}} + B$$

When $t = 0, x = 0$

$0 = 0 - 4 \times 4^{\frac{3}{2}} + B$

$B = 4 \times 4^{\frac{3}{2}} = 4 \times 8 = 32$

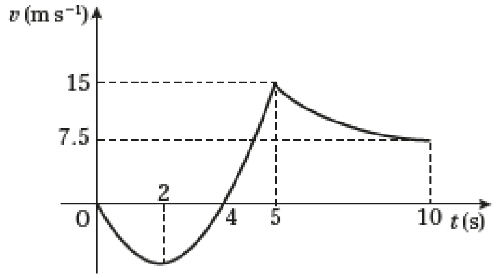
Hence $x = 30t - 4(t+4)^{\frac{3}{2}} + 32$

When $t = 21$

$x = 30 \times 21 - 4(25)^{\frac{3}{2}} + 32 = 630 - 500 + 32 = 162$

The distance of P from O when P comes to instantaneous rest is 162 m.

5 a



In the interval $0 \leqslant t \leqslant 5$, the graph is part of a parabola which meets the t-axis at the origin and where $t = 4$.
In the interval $5 < t \leqslant 10$, the graph is a segment of a hyperbola joining $(5, 15)$ to $(10, 7.5)$.

b The set of values of t for which the acceleration is positive is $2 < t < 5$.

The acceleration is positive when the velocity–time graph has a positive gradient. By the symmetry of a parabola, the graph has a minimum when $t = 2$ and the set of values of t for which the gradient is positive can be written down by inspecting the graph.

c

$$\int_0^4 3t(t-4)\,\mathrm{d}t = \int_0^4 (3t^2 - 12t)\,\mathrm{d}t$$

$$= \left[t^3 - 6t^2\right]_0^4$$

$$= (64 - 96) - 0 = -32$$

$$\int_4^5 3t(t-4)\,\mathrm{d}t = \int_4^5 (3t^2 - 12t)\,\mathrm{d}t$$

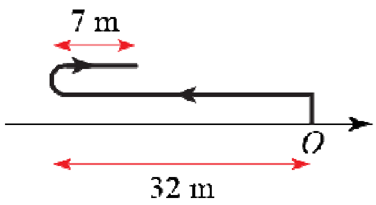
$$= \left[t^3 - 6t^2\right]_4^5$$

$$= (125 - 150) - (64 - 96)$$

$$= 7$$

The distance travelled by P in the interval $0 \leqslant t \leqslant 5$ is $(32 + 7)\text{m} = 39\,\text{m}$.

Taking the direction of v increasing as positive, for the first 4 seconds the particle travels 32 m in the negative direction. In the next second, it travels 7 m in the positive direction. So in 5 seconds, it travels a total of (32 + 7)m ending at a point which is (32–7)m from O in the negative direction.



d For $t > 5$

$$x = \int v\,\mathrm{d}t = \int 75t^{-1}\,\mathrm{d}t$$

$$= 75\ln t + A$$

At time $t = 5$, the particle is $(32 - 7)\text{m} = 25\text{m}$ from O in the negative direction.

So when $t = 5, x = -25$

$$-25 = 75\ln 5 + A \Rightarrow A = -75\ln 5 - 25$$

Using the law of logarithms

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right),$$

$$75\ln t - 75\ln 5 = 75(\ln t - \ln 5) = 75\ln\left(\frac{t}{5}\right).$$

Hence

$$x = 75\ln t - 75\ln 5 - 25 = 75\ln\left(\frac{t}{5}\right) - 25$$

At $x = 0$

$$0 = 75\ln\left(\frac{t}{5}\right) - 25 \Rightarrow \ln\left(\frac{t}{5}\right) = \frac{1}{3}$$

You solve this equation for t by taking exponentials of both sides of the equation and using $e^{\ln\left(\frac{t}{5}\right)} = \frac{t}{5}$.

$$\frac{t}{5} = e^{\frac{1}{3}} \Rightarrow t = 5e^{\frac{1}{3}} = 6.98\ (3\text{ s.f.})$$

6 $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 4x$

When the acceleration is a function of the displacement, x metres, you write $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$ and integrate both sides of the equation with respect to x.

$\frac{1}{2}v^2 = \int 4x\,\mathrm{d}x = 2x^2 + A$

$v^2 = 4x^2 + B$, where $B = 2A$

At $x = 2, v = 4$

$16 = 16 + B \Rightarrow B = 0$

Even when, as here, the constant of integration is 0, it is essential for you to show how this follows from the information given in the question to gain full marks.

Hence

$v^2 = 4x^2$

7 $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 1 - \frac{4}{x^2} = 1 - 4x^{-2}$

$\frac{1}{2}v^2 = \int (1 - 4x^{-2})\mathrm{d}x$

$= x - \frac{4x^{-1}}{-1} + A = x + \frac{4}{x} + A$

$v^2 = 2x + \frac{8}{x} + B$, where $B = 2A$

Multiplying the equation $\frac{1}{2}v^2 = 2x + \frac{4}{x} + A$ throughout by 2.

Twice one arbitrary constant is another arbitrary constant.

At $x = 1, v = 3\sqrt{2}$

$18 = 2 + 8 + B \Rightarrow B = 8$

Hence $v^2 = 2x + \frac{8}{x} + 8$

At $x = \frac{3}{2}$

$v^2 = 2 \times \frac{3}{2} + 8 \times \frac{2}{3} + 8 = 11 + \frac{16}{3} = \frac{49}{3}$, as required.

You use the information that at $x = 1$, $v = 3\sqrt{2}$ to evaluate the constant of integration B. You then substitute $x = \frac{3}{2}$ into the resulting equation and show that $v^2 = \frac{49}{3}$.

8 $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 5 + 3\sin 3x$

$\frac{1}{2}v^2 = \int (5 + 3\sin 3x)\mathrm{d}x = 5x - \cos 3x + A$

$v^2 = 10x - 2\cos 3x + B$, where $B = 2A$

At $x = 0, v = 4$

$16 = 0 - 2 + B \Rightarrow B = 18$

You use the information that at $x = 0$, $v = 4$ to evaluate the constant of integration B. You then substitute $x = 6$ into the resulting equation and use your calculator to find v.

Hence $v^2 = 10x - 2\cos 3x + 18$

At $x = 6$

$v^2 = 60 - 2\cos 18 + 18 = 76.679...$

When calculus has been used, it is assumed that all angles are measured in radians and you must make sure that your calculator is in the correct mode.

$v = \sqrt{76.679...} = 8.756...$

The speed of P at $x = 6$ is 8.76 m s^{-1} (3 s.f.)

9 a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4k^2}{(x+1)^2} = 4k^2(x+1)^{-2}$

$\frac{1}{2}v^2 = \int 4k^2(x+1)^{-2}\,dx = \frac{4k^2(x+1)^{-1}}{-1} + A$

$v^2 = B - \frac{8k^2}{x+1}$, where $B = 2A$

At $x = 1, v = 0$

$0 = B - \frac{8k^2}{2} \Rightarrow B = 4k^2$

Hence $v^2 = 4k^2 - \frac{8k^2}{x+1} = 4k^2\left(1 - \frac{2}{x+1}\right)$

b $v = 2k\sqrt{\left(1 - \frac{2}{x+1}\right)}$

As P is moving on the positive x-axis in the direction of x increasing, you need not consider the possibility of a negative square root.

As x is positive, $1 - \frac{1}{x+1} < 1$

As x is positive, $\frac{1}{1+x}$ is positive and one minus a positive number must be less than one.

Hence $v < 2k$ and v cannot exceed $2k$.

10 a At the maximum value of v, $\frac{dv}{dt} = 0$.

As $a = \frac{dv}{dt}$, the maximum speed of P occurs when $a = \frac{1}{12}(30 - x) = 0 \Rightarrow x = 30$.

b $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{12}(30 - x)$

$\frac{1}{2}v^2 = \int \frac{1}{12}(30 - x)dx = \int\left(\frac{5}{2} - \frac{x}{12}\right)dx$

$= \frac{5x}{2} - \frac{x^2}{24} + A$

$v^2 = 5x - \frac{x^2}{12} + B$, where $B = 2A$

Multiplying the equation $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{24} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 30, v = 10$

$100 = 5 \times 30 - \frac{900}{12} + B$

$B = 100 + \frac{900}{12} - 150 = 25$

Hence $v^2 = 5x - \frac{x^2}{12} + 25$

An alternative form of this answer, completing the square, is $v^2 = 100 - \frac{1}{12}(30 - x)^2$. This confirms that the speed has a maximum at $x = 30$.

11 a $u = 0, a = 20, v = 6, s = ?$

$v^2 = u^2 + 2as$

$36 = 0 + 2 \times 20 \times s$

$s = \frac{36}{40} = 0.9$

The model in part **a** is that of constant acceleration, which you studied in module M1. The specification for M3 includes 'a knowledge of the specifications for M1 and M2 and their prerequisites and associated formulae ... is assumed and may be tested'.

For the first model, the distance moved by P while accelerating from rest to 6 m s^{-1} is 0.9 m.

b $a = p - qx$

At $x = 0, a = 20$

$20 = p - 0 \Rightarrow p = 20$

Hence $a = 20 - qx$

At $x = 2, a = 12$

$12 = 20 - 2q \Rightarrow q = \frac{20-12}{2} = 4$

$p = 20, q = 4,$ as required.

The initial acceleration is 20 m s^{-2}. This applies to all parts of the question. Additionally in part **b**, you are given that $a = 12$ when $x = 2$. The two conditions enable you to find the two unknowns p and q.

c $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 20 - 4x$

$\frac{1}{2}v^2 = \int (20 - 4x)dx = 20x - 2x^2 + A$

$v^2 = 40x - 4x^2 + B,$ where $B = 2A$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

Hence $v^2 = 40x - 4x^2$

When $v = 6$

$36 = 40x - 4x^2 \Rightarrow 4x^2 - 40x + 36 = 0$

Divide this equation throughout by 4 and factorise.

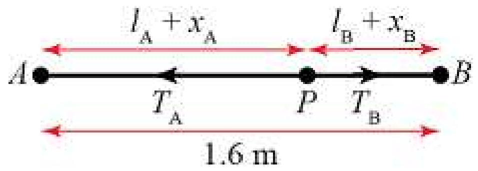
$x^2 - 10x + 9 = (x-1)(x-9) = 0$

$x = 1, 9$

The distance moved by P in first attaining a speed of 6 m s^{-1} is 1 m.

Comparing this with result in part **a**, the revised model predicts that P moves a little further before reaching the speed of 6 m s^{-1}.

12



$AB = 1.6$ m

For spring AP: $l = l_A = 0.8$ m, $\lambda = \lambda_A = 24$ N, $x = x_A$, $T = T_A$
For spring PB: $l = l_B = 0.4$ m, $\lambda = \lambda_A = 20$ N, $x = x_B$, $T = T_B$

$l_A + l_B + x_A + x_B = 1.6$

$x_A + x_B = 1.6 - 0.8 - 0.4 = 0.4$
$x_B = 0.4 - x_A$

Since P is in equilibrium, $T_A = T_B$

a Using Hooke's law for each spring:

$$T = \frac{\lambda x}{l}$$

$$\frac{\lambda_A x_A}{l_A} = \frac{\lambda_B x_B}{l_B}$$

$$\frac{24x_A}{0.8} = \frac{20x_B}{0.4}$$

$30x_A = 50x_B$ substituting for x_B

$$3x_A = 5(0.4 - x_A)$$

$$8x_A = 2$$

$$x_A = 0.25$$

$$AP = l_A + x_A$$

$$AP = 0.8 + 0.25 = 1.05$$

The distance AP is 1.05 m.

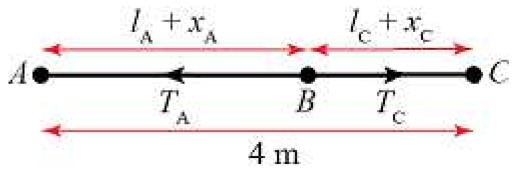
b Substituting the value for x_A into the expression for T_A

$$T_A = \frac{\lambda_A x_A}{l_A}$$

$$T_A = \frac{24 \times 0.25}{0.8} = 7.5$$

Since P is in equilibrium, $T_A = T_B$
The tension in each spring is 7.5 N

13



AC = 4 m

For spring AB: $l = l_A = 1.5$ m, $\lambda = \lambda_A = 20$ N, $x = x_A$, $T = T_A$
For spring BC: $l = l_C = 0.75$ m, $\lambda = \lambda_A = 15$ N, $x = x_C$, $T = T_C$

$l_A + l_C + x_A + x_C = 4$

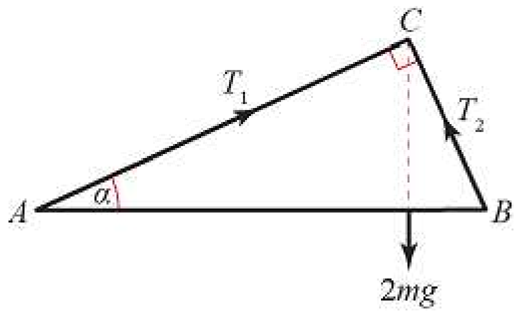
$x_A + x_C = 4 - 1.5 - 0.75 = 1.75$
$x_C = 1.75 - x_A$

Since system is in equilibrium, $T_A = T_C$
Using Hooke's law for each spring:

$$T = \frac{\lambda x}{l}$$
$$\frac{\lambda_A x_A}{l_A} = \frac{\lambda_C x_C}{l_C}$$
$$\frac{20x_A}{1.5} = \frac{15x_C}{0.75}$$
$$\frac{40x_A}{3} = 20x_C$$
$$2x_A = 3x_C \text{ substituting for } x_C$$
$$2x_A = 3(1.75 - x_A)$$
$$5x_A = 5.25$$
$$x_A = 1.05$$
$$AB = l_A + x_A$$
$$AB = 1.5 + 1.05 = 2.55$$
$$BC = 4 - AB$$
$$BC = 4 - 2.55 = 1.45$$

The distances *AB* and BC are 2.55 m and 1.45 m respectively.

14



a The line of action of the weight must pass through C which is not above the centre of the rod.

For three forces to be in equilibrium the lines of action of all three forces must pass through the same point. As the lines of action of both tensions pass through C, the line of action of the weight has to pass through C as well, so the rod cannot be uniform.

14 b $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha \frac{3}{5}, \cos\alpha = \frac{4}{5}$

Let the tension in AC be T_2 newtons and the tension in BC be T_2 newtons.

$R(\rightarrow) \quad T_1\cos\alpha = T_2\sin\alpha$

$\frac{4}{5}T_1 = \frac{3}{5}T_2 \Rightarrow T_1 = \frac{3}{4}T_2$

$R(\uparrow)\ T_1\sin\alpha + T_2\cos\alpha = 2mg$

You substitute $T_1 = \frac{3}{4}T_2$ and the values of $\sin\alpha$ and $\cos\alpha$ into this equation and solve for T_2

$$\frac{3}{4}T_2 \times \frac{3}{5} + T_2 \times \frac{4}{5} = 2\,mg$$

$$\left(\frac{9}{20} + \frac{4}{5}\right)T_2 = \frac{5}{4}T_2 = 2\,mg$$

$$T_2 = \frac{8}{5}\,mg$$

The tension in BC is $\frac{8}{5}mg$, as required.

$T_1 = \frac{3}{4}T_2 = \frac{3}{4} \times \frac{8}{5}mg = \frac{6}{5}mg$

The tension in AC is $\frac{6}{5}mg$.

c $BC = AB\sin\alpha = 2a \times \frac{3}{5} = \frac{6}{5}a$

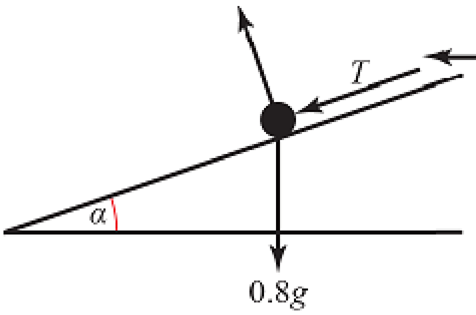
You find the length of BC by trigonometry. Then the extension of the elastic string BC is $\frac{6}{5}a - a = \frac{1}{5}a$

For BC

Hooke's law $T_2 = \frac{\lambda x}{l}$

$\frac{8}{5}mg = \frac{kmg \times \frac{1}{5}a}{a} \Rightarrow k = 8$

15



Initially the spring is in compression and the force of the spring on the particle is acting down the plane.

Let the thrust in the spring be T newtons.

Hooke's law $T = \dfrac{\lambda x}{l}$

$$= \frac{20 \times 0.4}{2} = 4$$

The compression is $(2 - 1.6)$ m $= 0.4$ m

R(↗) $\mathbf{F} = m\mathbf{a}$

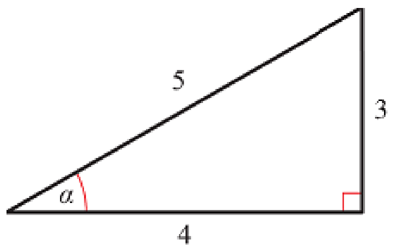
$$mg \sin\alpha + T = ma$$

$$0.8 \times 9.8 \times \frac{3}{5} + 4 = 0.8a$$

$$0.8a = 8.704$$

$$a = 10.88$$

When you know $\tan\alpha$ you can draw a triangle to find $\cos\alpha$ and $\sin\alpha$.



$\tan\alpha = \dfrac{3}{4}$

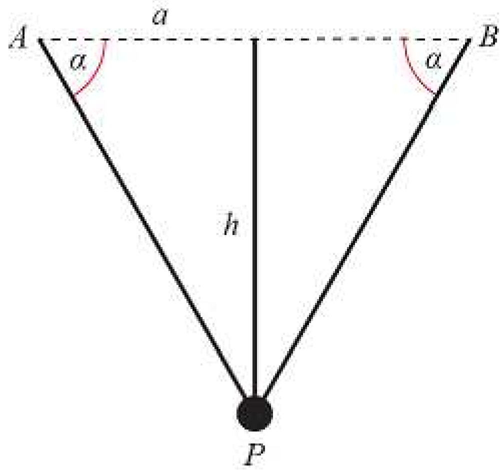
$\sin\alpha = \dfrac{3}{5}$

$\cos\alpha = \dfrac{4}{5}$

The initial acceleration of the particle is 11 m s^{-1} (2 s.f.)

As you have used an approximate value of g, you should round your answer to a sensible accuracy. Either 2 or 3 significant figures is acceptable.

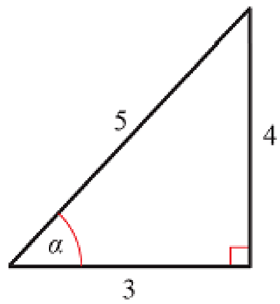
16



When P comes instantaneously to rest, it is not in equilibrium and so the question cannot easily be solved by resolving. It is a common error to attempt the solution of this, and similar questions, by resolving.

$\tan\alpha = \frac{4}{3} \Rightarrow \cos\alpha = \frac{3}{5}$

When you know $\tan\alpha$ you can draw a triangle to find $\cos\alpha$.



$\tan\alpha = \frac{4}{3}$

$\cos\alpha = \frac{3}{5}$

Let the distance fallen by P be h.

$$h = a\tan\alpha = \frac{4a}{3}$$

$$AP^2 = h^2 + a^2 = \left(\frac{4a}{3}\right)^2 + a^2 = \frac{25a^2}{9}$$

$$AP = \frac{5a}{3}$$

When P first comes to rest the energy stored in one string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{\lambda\left(\frac{2a}{3}\right)^2}{2a} = \frac{2\lambda a}{9}$$

The extension in one string is

$$\text{AP} - \text{natural length} = \frac{5a}{3} - a$$

$$= \frac{2a}{3}$$

When P first comes to rest the potential energy lost is given by

$$mgh = mg \times \frac{4}{3}a$$

Conservation of energy

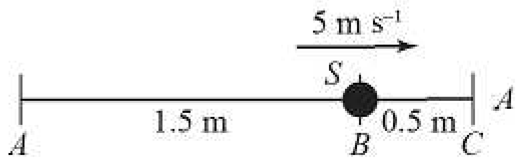
Elastic energy gained = potential energy lost

Initially P is at rest and, when it has fallen $\frac{5a}{3}$, it is at rest again. So there is no change in kinetic energy. Elastic energy is gained by both strings and potential energy is lost by the particle.

$$\frac{4\lambda a}{9} = \frac{4mga}{3}$$

$$\lambda = \frac{4mga}{3} \times \frac{9}{4a} = 3\,mg$$

17



a Let $AC = 2$ m. When S is at C, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{20\times(0.5)^2}{2\times1.5} = \frac{5}{3}\text{ J}$$

Let the speed of S at C be v m s^{-1}
Conservation of energy

Kinetic energy lost = elastic potential energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{5}{3}$$

$$\frac{1}{2}\times0.2\times5^2 - \frac{1}{2}\times0.2v^2 = \frac{5}{3}$$

$$0.1v^2 = 0.1\times25 - \frac{5}{3} = \frac{5}{6}$$

$$v^2 = \frac{25}{3} \Rightarrow V = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3} \approx 2.886\ldots$$

The exact answer $\frac{5\sqrt{3}}{3}$ m s^{-1} is also accepted.

The speed of S when $AS = 2$ m is 2.89 m s^{-1} (3 s.f.)

b Let the extension of the string immediately before the string breaks be x m.
When the extension in the string is x m, the elastic energy stored in the string is given by

To find the tension in the string when the speed of S is 1.5 m s^{-1}, you first need to find the extension of the string at this speed. The extension is found using conservation of energy.

$$E = \frac{\lambda x^2}{2l} = \frac{20x^2}{3}$$

Conservation of energy
Kinetic energy lost = elastic energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{20x^2}{3}$$

$$\frac{1}{2}\times0.2\times5^2 - \frac{1}{2}\times0.2\times1.5^2 = \frac{20x^2}{3}$$

$$\frac{20x^2}{3} = 2.275 \Rightarrow x^2 = 0.34125$$

$$x = \sqrt{(0.34125)}$$

Hooke's law

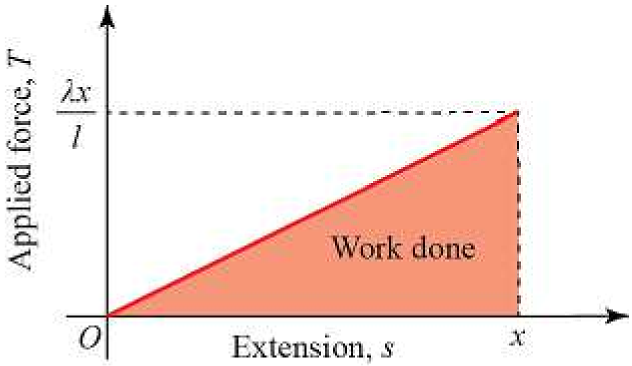
$$T = \frac{\lambda x}{l} = \frac{20\sqrt{(0.34125)}}{1.5} = 7.788\ldots$$

The tension in the string immediately before the string breaks is 7.79 N (3 s.f.)

18 Due to equivalence of work and energy:
energy stored = work done in stretching the string.
Work done in stretching the string is given by the area under the line (see graph):

$$\text{energy stored} = \frac{1}{2}x\left(\frac{\lambda x}{l}\right)$$

$$\text{energy stored} = \frac{\lambda x^2}{2l}$$



At equilibrium, the tension in the spring, $T = mg$
Using Hooke's law:

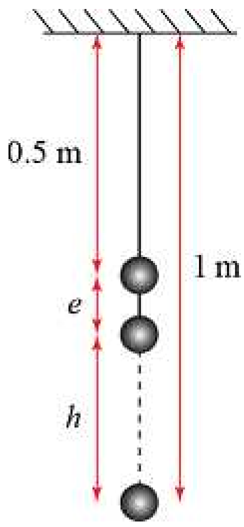
$$T = \frac{\lambda x}{l} = mg$$

$$x = \frac{lmg}{\lambda}$$

$$\text{so energy stored} = \frac{\lambda}{2l}\left(\frac{lmg}{\lambda}\right)^2$$

$$\text{energy stored} = \frac{m^2g^2l}{2\lambda} \quad \text{as required.}$$

19 $l = 0.5$ m, $\lambda = 20$ N, $m = 0.5$ kg



Due to the equivalence of work and energy:
work done in stretching the string
= energy stored when total length is 1.0 m – total energy stored at equilibrium length

When the string is stretched to a total length of 1.0 m, $x = 1.0 - 0.5 = 0.5$ m

and energy stored in the string at this length $= \dfrac{\lambda x^2}{2l}$

When the string is at equilibrium, the tension, $T = mg$
Let the extension at this point be e

Using Hooke's law:

$$T = \frac{\lambda x}{l} = mg$$

$$mg = \frac{\lambda e}{l}$$

$$e = \frac{mgl}{\lambda} = \frac{0.5 \times 0.5 \times 9.8}{20} = 0.1225$$

However, in this position there is also additional gravitational potential energy as particle is further above the ground.
gravitational potential energy $= mgh = mg(x - e)$

So work done in stretching the string from equilibrium to 1.0 m:

work done = final EPE – initial (EPE + PE)

$$\text{work done} = \frac{\lambda x^2}{2l} - \left(\frac{\lambda e^2}{2l} + mg(x - e)\right)$$

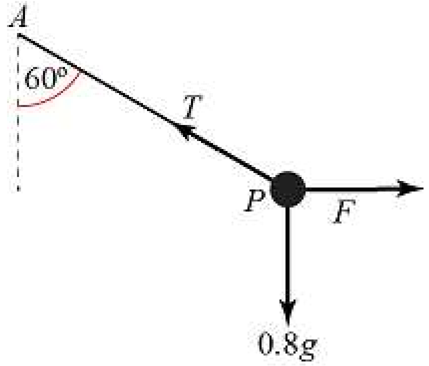
$$\text{work done} = \frac{\lambda}{2l}(x^2 - e^2) - mg(x - e)$$

$$\text{work done} = \frac{20}{2 \times 0.5}(0.5^2 - 0.1225^2) - 0.5 \times 9.8(0.5 - 0.1225)$$

$$\text{work done} = 20(0.25 - 0.01500...) - 4.9(0.5 - 0.1225) = 2.8501...$$

The work done in stretching the string is 2.85 J (3 s.f.)

20 a



R(↑) $T\cos 60^\circ = 0.8g$

$\frac{1}{2}T = 0.8g \Rightarrow T = 1.6g$

Resolving vertically gives you the tension in the string.

R(←) $F = T\cos 30^\circ = 1.6g \times \frac{\sqrt{3}}{2}$

$= 13.579$

$= 14 \text{ (2 s.f.)}$

Substituting for the tension into the equation obtained by resolving horizontally gives the value of F.

b Hooke's law $T = \frac{\lambda x}{l}$

$1.6g = \frac{24x}{1.2}$

$x = \frac{1.6g \times 1.2}{24} = 0.784$

Substituting for the tension into Hooke's Law gives you an equation for the extension.

The extension of the string is 0.78 m (2 s.f.)

c The elastic energy sotred in the string is given by

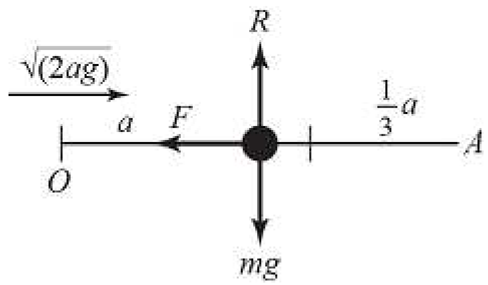
$E = \frac{\lambda x^2}{2l}$

You need to remember the formula for the energy stored in an elastic string.

$= \frac{24 \times (0.784)^2}{2 \times 1.2} = 6.14656$

The elastic energy stored in the string is 6.1 J (2 s.f.)

21



a At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{3.6\,mg \times (\frac{1}{3}a)^2}{2a}$$

$$= 0.2mga$$

At A, the extension of the string is $\frac{4}{3}a - a = \frac{1}{3}a$

b The total energy lost is

$$\frac{1}{2}mu^2 - 0.2mga = \frac{1}{2} \times 2mga - 0.2mga$$

$$= 0.8mga$$

As P is at rest at A, then net loss of energy is the loss in kinetic energy minus the gain in elastic energy.

At any point in the motion

R(↑) $R = mg$

The friction is given by

$F = \mu R = \mu mg$

By the work–energy principle

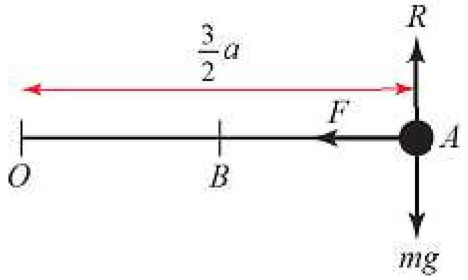
$$0.8mga = \mu mg \times \frac{4}{3}a$$

$$\mu = 0.8 \times \frac{3}{4} = 0.6$$

By the work–energy principle, the net loss in energy is equal to the work done by friction. You find the work done by friction by multiplying the magnitude of the friction, μmg, by the distance the particle moves, $\frac{4}{3}a$

This gives you an equation in μ, which you solve.

22



At any point in the motion

R(↑) $R = mg$

The friction is given by

$$F = \mu R = \frac{2}{3}mg$$

At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{4mg \times (\frac{1}{2}a)^2}{2a}$$

$$= \frac{1}{2}mga$$

At A, the extension of the string is $\frac{3}{2}a - a = \frac{1}{2}a$

By the work–energy principle

$$\frac{1}{2}mga = \frac{2}{3}mg \times AB$$

$$AB = \frac{3}{4}a$$

When P comes to rest, as $OB < a$, the string is slack so all of the elastic energy has been lost. This lost energy must equal the work done by friction, which is the magnitude of the friction, $\frac{2}{3}mg$, multiplied by the distance moved by P, which is AB.

23 $l = 1.0$ m, $\lambda = 75$ N, $m = 5$ kg, $x = 1.5 - 1.0 = 0.5$ m

Energy stored in the spring is transferred to the kinetic energy of the particle.

By the conservation of energy

$$\frac{\lambda x^2}{2l} = \frac{1}{2}mv^2$$

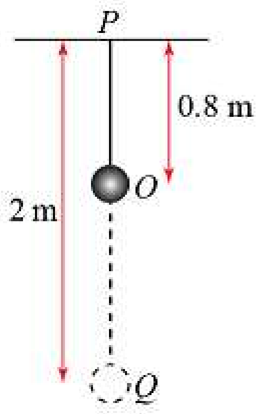
$$\frac{\lambda x^2}{l} = mv^2$$

$$v^2 = \frac{\lambda x^2}{ml}$$

$$v^2 = \frac{75 \times 0.5^2}{5 \times 1} = \frac{15}{4}$$

$$v = \frac{\sqrt{15}}{2}$$ as required.

24 $l = 0.8$ m, $\lambda = 15$ N, $m = 0.5$ kg, $x = 2 - 0.8 = 1.2$ m



Energy stored in string when stretched to total length 2 m

$$E = \frac{\lambda x^2}{2l}$$

$$E = \frac{15 \times 1.2^2}{2 \times 0.8} = 13.5$$

As particle moves upwards, this is converted into gravitational potential energy and kinetic energy.

a When string first becomes slack, the particle is $h = 1.2$ m above initial position.
Initial elastic potential energy = final potential energy + final kinetic energy

$$E = mgh + \frac{1}{2}mv^2$$

$$E = m\left(gh + \frac{1}{2}v^2\right)$$

$$\frac{1}{2}v^2 = \frac{E}{m} - gh$$

$$v = \sqrt{2\left(\frac{E}{m} - gh\right)}$$

$$v = \sqrt{2\left(\frac{13.5}{0.5} - (9.8 \times 1.2)\right)} = 5.5208...$$

When the string first becomes slack, the particle is travelling at 5.5 ms^{-1} (2 s.f.)

b When particle reaches P, $h = 2$ m

$$v = \sqrt{2\left(\frac{13.5}{0.5} - (9.8 \times 2)\right)} = 3.8470...$$

When the particle reaches P, it is travelling at 3.8 m s^{-1} (2 s.f.)

25 a When P comes to rest for the first time, let the extension of the string be x m

Conservation of energy

elastic energy gained = potential energy lost

$$\frac{\lambda x^2}{2l} = mgh$$
$$\frac{58.8x^2}{8} = 0.5 \times 9.8 \times (4+x)$$
$$7.5x^2 = 19.6 + 4.9x$$
$$3x^2 - 2x - 8 = 0$$
$$(x-2)(3x+4) = 0$$
$$x = 2$$

Divide this equation throughout by 2.45 and rearrange the terms. If you cannot see this simplification, you can use the quadratic formula but you would be expected to obtain an exact answer.

For the string to have elastic energy, it has to be stretched so you can ignore the negative solution $-\frac{4}{3}$

The distance fallen by P is (4 + 2) m = 6 m

b P will first become slack when it has moved 3 m vertically.

Let the velocity at this point be v m s^{-1}

Conservation of energy

kinetic energy gained + potential energy gained = elastic energy lost

$$\frac{1}{2}mv^2 + mgh = \frac{\lambda x^2}{2l}$$
$$\frac{1}{2}0.5v^2 + 0.5 \times 9.8 \times 3 = \frac{58.8 \times 3^2}{8}$$
$$0.25v^2 = 14.7 = 66.15$$
$$v^2 = \frac{66.15 - 14.7}{0.25} = 205.8$$
$$v = \sqrt{(205.8)} = 14.345\ldots$$

Initially P is at rest and then rise 3 m. So both kinetic and potential energy are gained. Initially the string is stretched but, after rising 3 m, it is slack. So elastic energy is lost. By Conservation of energy, the net gain of kinetic and potential energies must equal the elastic energy lost.

The speed of the particle when the string first becomes slack is 14 m s^{-1} (2 s.f.)

26 a $\mathbf{F} = m\mathbf{a}$

$2e^{-0.1x} = 2.5a$

$a = \frac{2}{2.5}e^{-0.1x} = 0.8e^{-0.1x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.8e^{-0.1x}$

Using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

$\frac{1}{2}v^2 = \int 0.8e^{-0.1x}\,dx = \frac{0.8e^{-0.1x}}{-0.1} + A$

$= -8e^{-0.1x} + A$

$v^2 = B - 16e^{-0.1x}$, where $B = 2A$

Twice one arbitrary constant is another arbitrary constant.

When $x = 0, v = 2$

$4 = B - 16 \Rightarrow B = 20$

At $x = 0, e^{-0.1x} = e^0 = 1$.

Hence $v^2 = 20 - 16e^{-0.1x}$

b When $v = 4$

$16 = 20 - 16e^{-0.1x}$

$e^{-0.1x} = \frac{20-16}{16} = \frac{1}{4}$

Take logarithms of both sides of this equation and use $\ln(e^{-0.1x}) = -0.1x$.

$-0.1x = \ln\left(\frac{1}{4}\right) = -\ln 4$

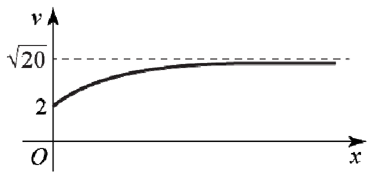
$x = 10\ln 4$

26 c For all x, $e^{-0.1x} > 0$

So $v^2 = 20 - 16e^{-0.1x} < 20$

and hence $v < \sqrt{20}$

The speed of P does not exceed $\sqrt{20}\ \text{m s}^{-1}$.



This sketch, of v against x, shows that as x increases, the velocity of P gets closer to $\sqrt{20}\ \text{m s}^{-1}$ but never reaches it. $\sqrt{20}\ \text{m s}^{-1}$ is the **terminal** or **limiting speed** of P.

27 a $\mathbf{F} = m\mathbf{a}$

$$\frac{1}{10}x(4-3x) = 0.2a$$

$$a = \frac{1}{0.2\times 10}x(4-3x) = \frac{1}{2}x(4-3x) = 2x - \frac{3x^2}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 2x - \frac{3x^2}{2}$$

Integrate both sides of this equation with respect to x. Remember to include a constant of integration.

$$\frac{1}{2}v^2 = \int\left(2x - \frac{3x^2}{2}\right)\mathrm{d}x = x^2 - \frac{x^3}{2} + A$$

$$v^2 = 2x^2 - x^3 + B, \text{where } B = 2A$$

At $x = 6, v = 0$

The car comes to instantaneous rest when $x = 6$. So $v = 0$ at $x = 6$.

$0 = 2\times 36 - 216 + B \Rightarrow B = 144$

Hence $v^2 = 2x^2 - x^3 + 144$

b When $x = 0$

$v^2 = 144 \Rightarrow v = \pm 12$

Both signs are possible as the car could pass through O in either direction when $t = 0$. However, in either case, the speed of the car, which is the magnitude of the velocity, is 12 m s^{-1}.

The initial speed of the car is 12 m s^{-1}.

28 a $\mathbf{F} = m\mathbf{a}$

$$\frac{48\,000}{(t+2)^2} = 800a$$

$$a = \frac{dv}{dt} = \frac{60}{(t+2)^2} = 60(t+2)^{-2}$$

$$v = \int 60(t+2)^{-2}\,dt = \frac{60(t+2)^{-1}}{-1} + A$$

$$= A - \frac{60}{t+2}$$

When $t = 0, v = 0$

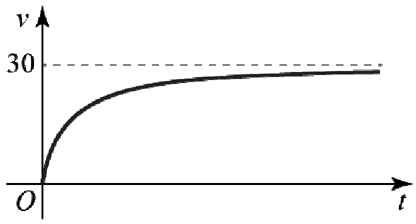
$$0 = A - \frac{60}{2} \Rightarrow A = 30$$

Hence $v = 30 - \frac{60}{t+2}$

As $t \to \infty, \frac{60}{t+2} \to 0$ and $v \to 30$

As t increases, the car approaches a limiting speed of 30 m s^{-1}.

When the acceleration is a function of t, the velocity can be found by writing $a = \frac{dv}{dt}$ and integrating with respect to t.



As the value of t increases, the value of $\frac{60}{t+2}$ decreases and so $30 - \frac{60}{t+2}$ gets closer to 30. The graph of v against t approaches $v = 30$ as an asymptote.

b The distance moved in the first 6 s is given by

$$x = \int_0^6 \left(30 - \frac{60}{t+2}\right) dt$$

$$= \left[30t - 60\ln(t+2)\right]_0^6$$

$$= (180 - 60\ln 8) - (0 - 60\ln 2)$$

$$= 180 - 60\ln 2^3 + 60\ln 2$$

$$= 180 - 180\ln 2 + 60\ln 2$$

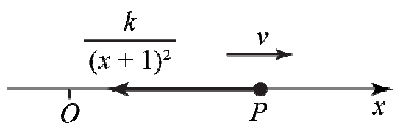
$$= 180 - 120\ln 2$$

The car is always travelling in the same direction. It does not turn round and so the distance moved in the interval $0 \leqslant t \leqslant 6$ can be found by evaluating the definite integral $\int_0^6 v\,dt$.

The distance moved by the car in the first 6 s of its motion is (180 – 120 ln 2) m.

29 a $\mathbf{F} = m\mathbf{a}$

$$-\frac{k}{(x+1)^2}f = \frac{1}{3}a$$



The particle is moving along the positive x-axis and the force is directed toward O. So the force is in the direction of x decreasing and force has a negative sign in this equation.

$$a = -\frac{3k}{(x+1)^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3k(x+1)^{-2}$$

$$\frac{1}{2}v^2 = \frac{-3k(x+1)^{-1}}{-1} + A = \frac{3k}{x+1} + A$$

$$v^2 = \frac{6k}{x+1} + B, \text{where } B = 2A$$

At $x = 1, v = 4$

$$16 = \frac{6k}{2} + B \Rightarrow 3k + B = 16 \qquad \textbf{(1)}$$

At $x = 8, v = \sqrt{2}$

$$2 = \frac{6k}{9} + B \Rightarrow \frac{2}{3}k + B = 2 \qquad \textbf{(2)}$$

The information in the question gives you the values of v at two values of x and you use the information to obtain two simultaneous equations, which you solve.

(1) − (2)

$$3k - \frac{2}{3}k = \frac{7}{3}k = 14$$

$$k = 14 \times \frac{3}{7} = 6$$

b Substituting $k = 6$ into **(1)**

$$18 + B = 16 \Rightarrow B = -2$$

Hence $v^2 = \frac{36}{x+1} - 2$

To find the value of x for which P comes to rest, substitute $v = 0$ into this equation and solve for x.

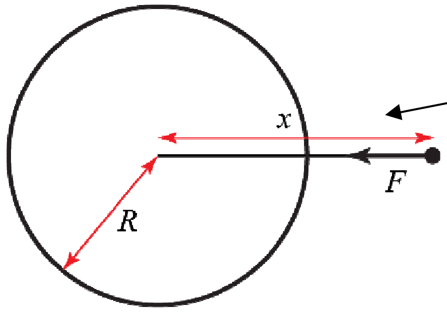
When $v = 0$

$$0 = \frac{36}{x+1} - 2 \Rightarrow 2(x+1) = 36$$

$$2x + 2 = 36 \Rightarrow x = \frac{36-2}{2} = 17$$

The distance of P from O when P first comes to instantaneous rest is 17 m.

30 a



The gravitational force is directed towards the centre of the Earth and so is in the direction of x decreasing.

As F is proportional to $\frac{1}{x^2}$, $F = -\frac{k}{x^2}$

At $x = R$, $F = -mg$

$-mg = -\frac{k}{R^2} \Rightarrow k = mgR^2$

You introduce a constant of proportionality k and use the fact, that the force due to gravity at the surface of the Earth is known to have magnitude mg, to find k.

Hence $F = -\frac{mgR^2}{x^2}$

The magnitude of the force is $\frac{mgR^2}{x^2}$, as required.

b $\mathbf{F} = m\mathbf{a}$

$-\frac{mgR^2}{x^2} = ma$

In this equation, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$a = -\frac{gR^2}{x^2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$

$\frac{1}{2}v^2 = -\int gR^2x^{-2}\ dx = -\frac{gR^2x^{-1}}{-1} + A$

$v^2 = \frac{2gR^2}{x} + B$, where $B = 2A$

At $x = R$, $v^2 = \frac{3}{2}gR$

The question gives the velocity of the particle as it is fired from the surface of the Earth. That is the velocity when $x = R$, the radius of the Earth.

$\frac{3}{2}gR = \frac{2gR^2}{R} + B \Rightarrow B = \frac{3}{2}gR - 2gR = -\frac{1}{2}gR$

Hence $v^2 = \frac{2gR^2}{x} - \frac{1}{2}gR$

When $x = 3R$

When the particle is at a height of $2R$ above the surface of the Earth, it is $2R + R = 3R$ from the centre of the Earth.

$v^2 = \frac{2gR^2}{3R} - \frac{1}{2}gR = \frac{gR}{6} \Rightarrow v = \sqrt{\left(\frac{gR}{6}\right)}$

The speed of the particle when it is $2R$ above the surface of the Earth is $\sqrt{\left(\frac{gR}{6}\right)}$.

31 a $\mathbf{F} = m\mathbf{a}$

$$-\frac{k}{x^2} = ma \quad \textbf{(1)}$$

$$a = -\frac{k}{mx^2}$$

At $x = R, a = -g$

$$-g = \frac{k}{mR^2}$$

$$k = mgR^2$$

Substituting $k = mgR^2$ into **(1)**

$$-\frac{mgR^2}{x^2} = ma$$

$$a = v\frac{dv}{dx} = -\frac{gR^2}{x^2}, \text{as required.}$$

The force is negative in equation **(1)** as the force on P due to gravity is directed towards the centre of the Earth and that is the direction of x decreasing.

You know that the acceleration due to gravity at the surface of the Earth is g and that the direction of the acceleration is towards the centre of the Earth. Substituting $a = -g$ into **(1)** gives you k in terms of m, g and R.

$a = v\frac{dv}{dx}$ is one of the alternative forms of the acceleration. $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$ and you must pick out the form of a which you need in any particular question.

b Separating the variables in the printed answer to part **a** and integrating

$$\int v\,dv = -\int \frac{gR^2}{x^2}dx = -\int gR^2x^{-2}dx$$

$$\frac{1}{2}v^2 = \frac{-gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{where } B = 2A$$

At $x = R, v = U$

$$U^2 = \frac{2gR^2}{R} + B \Rightarrow B = U^2 - 2gR$$

Hence $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$

When $v = 0, x = X$

$$0 = \frac{2gR^2}{X} + U^2 - 2gR$$

$$0 = 2gR^2 + U^2X - 2gRX$$

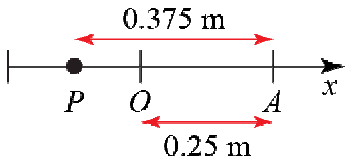
$$X(2gR - U^2) = 2gR^2$$

$$X = \frac{2gR^2}{2gR - U^2}$$

The projectile is fired from a point on the Earth's surface with speed U. This gives you that at $x = R, v = U$.

Multiply this equation throughout by X and then make X the subject of the formula.

32



The motion is simple harmonic with amplitude, a m, given by $a = 0.25$.

At P, $x = 0.25 - 0.375 = -0.125$

When $AP = 0.375$ m, P is 0.125 m from the centre of oscillation O. It is the other side of O from A and, if OA is taken as the direction of x increasing, the displacement of P is 0.125 m.

The Period is 2 s.

Hence $T = \frac{2\pi}{\omega} = 2 \Rightarrow \omega = \pi$

$$x = a\cos\omega t$$
$$-0.125 = 0.25\cos\pi t$$
$$\cos\pi t = -\frac{1}{2}$$

If the time, t seconds, is measured from the time when the velocity is zero, that is when the distance of P from O is the amplitude, then $x = a\cos\omega t$ is the appropriate formula connecting the displacement with the time.

The smallest positive value of t is given by

$$\pi t = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$
$$t = \frac{2}{3}$$

33 a $T = \frac{2\pi}{\omega} = \pi \Rightarrow \omega = 2$

When $x = 0.5, v = 2.4$

$$v^2 = \omega^2(a^2 - x^2)$$
$$2.4^2 = 2^2(a^2 - 0.5^2)$$
$$a^2 - 0.25 = \frac{2.4^2}{2^2} = 1.44$$
$$a^2 = 1.69 \Rightarrow a = 1.3$$

The amplitude of the motion is 1.3 m.

b The maximum speed is given by

$v = \omega a = 2 \times 1.3 = 2.6$

The maximum speed of P during its motion is 2.6 m s^{-1}.

As $v^2 = \omega^2(a^2 - x^2)$ and x^2 is positive for all x, the greatest value of v^2 is at $x = 0$. So the greatest value of v^2 is $\omega^2 a^2$ and the greatest value of the speed is ωa.

c The maximum magnitude of the acceleration is given by

$|\ddot{x}| = |\omega^2 a| = 4 \times 1.3 = 5.2$

The maximum magnitude of the acceleration is 5.2 m s^{-2}.

The acceleration is given by $\ddot{x} = \omega^2 x$ and this has the greatest size when x is the amplitude.

33 d $x = a\cos\omega t$

Differentiating with respect to t

$$\dot{x} = -a\omega\sin\omega t$$

$$|\dot{x}| = |a\omega\sin\omega t|$$

$$2.4 = 1.3\times 2\sin 2t_1 = 2.6\sin 2t_1$$

$$\sin 2t_1 = \frac{12}{13}$$

$$2t_1 = \arcsin\left(\frac{12}{13}\right) = 1.176...$$

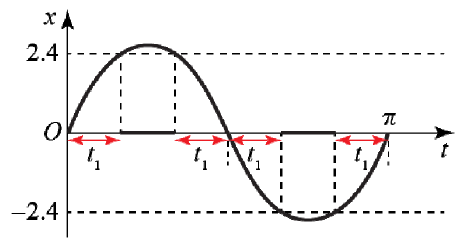
$$t_1 = 0.588...$$

The required time is given by

$$T' = \pi - 4t_1 = \pi - 4\times 0.588...$$

$$= 0.789...$$

The time for which the speed is greater than 2.4 m s^{-1} is 0.79 s (2 d.p.).



This diagram illustrates that if t_1 is the smallest positive solution of $2.6\sin 2t_1 = 2.4$, the time for which the speed is greater than 2.4 is $\pi - 4t_1$.

34 a At C $v^2 = \omega^2(a^2 - x^2)$

$$0^2 = \omega^2(a^2 - 1.2^2) \Rightarrow a = 1.2$$

At A $v^2 = \omega^2(a^2 - x^2)$

$$\left(\frac{3}{10}\sqrt{3}\right)^2 = \omega^2(1.2^2 - 0.6^2)$$

$$\frac{27}{100} = \omega^2\times 1.08$$

$$\omega^2 = \frac{27}{108} = \frac{1}{4} \Rightarrow \omega = \frac{1}{2}$$

You show that the information is the question is consistent with SHM by taking the information you have been given about two of the points and using it to find the values of a and ω. You then confirm these values are correct using the information about the third point. As the information about C gives you a directly, it is sensible to start with that point.

Checking $a = 1.2$ and $\omega = \frac{1}{2}$ at B

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{1}{4}(1.2^2 - 0.8^2) = 0.2 = \frac{1}{5}$$

$$v = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{1}{5}\sqrt{5}$$

Using $a = 1.2$ and $\omega = \frac{1}{2}$, you find the speed of P at B.

This calculation confirms the speed of P given in the question and you deduce that the information is consistent with P performing simple harmonic motion.

This is consistent with the information in the question. The information is consistent with P performing SHM with centre O.

b At O, $x = 0$. Using $v^2 = \omega^2(a^2 - x^2)$

$$= \frac{1}{4}(1.2^2 - 0^2) = 0.36$$

$$v = \sqrt{0.36} = 0.6$$

The speed of P at O is 0.6 m s^{-1}, as required.

34 c At A $\ddot{x} = -\omega^2 x = -\frac{1}{4} \times 0.6 = -0.15$

The magnitude of the acceleration at A is 0.15 m s^{-2}.

d At A $x = a\sin\omega t$

In this question, as you need to find the difference between the times at which P is at A and B, it does not matter which of the formulae $x = a\sin\omega t$ or $x = a\cos\omega t$ you use. If you use the formula with cosine, you obtain $\frac{\pi}{3}$ s and 1.459455... s as the times. The difference between these times is again 0.412 (3 s.f).

$0.6 = 1.2\sin\frac{1}{2}t_1 \Rightarrow \sin\frac{1}{2}t_1 = \frac{1}{2}$

$\frac{1}{2}t_1 = \frac{\pi}{6} \Rightarrow t_1 = \frac{\pi}{3}$

At B $x = a\sin\omega t$

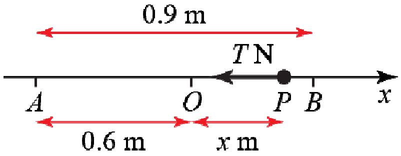
$0.8 = 1.2\sin\frac{1}{2}t_2 \Rightarrow \sin\frac{1}{2}t_2 = \frac{2}{3}$

$\frac{1}{2}t_2 = 0.729\,727... \Rightarrow t_2 = 1.459\,455...$

$t_2 - t_1 = 1.459\,455... - \frac{\pi}{3} = 0.412\,257...$

The time taken to move directly from A to B is 0.412 s (3 s.f).

35 a



Let the piston be modelled by the particle P.
Let O be the point where $AO = 0.6$ m
When P is at a general point in its motion,

let $OP = x$ metres and the forces of the spring on P be T newtons.

Displacements in simple harmonic questions are usually measured from the centre of the motion. At the centre, the acceleration of the particle is zero and the forces on the particle are in equilibrium. In this question, the point of equilibrium, O, is where the spring is at its natural length. No horizontal forces will then be acting on the particle.

Hooke's Law

$T = \frac{\lambda x}{l} = \frac{48x}{0.6} = 80x$

R($\rightarrow$)$\mathbf{F} = m\mathbf{a}$

$-T = 0.2\ddot{x}$

When x is positive, the tension in the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

$-80x = 0.2\ddot{x}$

$\ddot{x} = -400x = -20^2 x$

To show that P is moving with simple harmonic motion, you have to show that, at a general point of its motion, the equation of motion of P has the form $\ddot{x} = -kx$, where k is a positive constant. In this case $k = \omega^2 = 100$.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 20$. The period, T seconds, is given by

$T = \frac{2\pi}{20} = \frac{\pi}{10}$ s, as required.

35 b $a = 0.3,\ \omega = 20$

The maximum speed is given by

$v = a\omega = 0.3 \times 20 = 6$

The maximum speed is $6\ \text{m s}^{-1}$.

c When the length of the spring is 0.75 m

$x = 0.75 - 0.6 = 0.15$

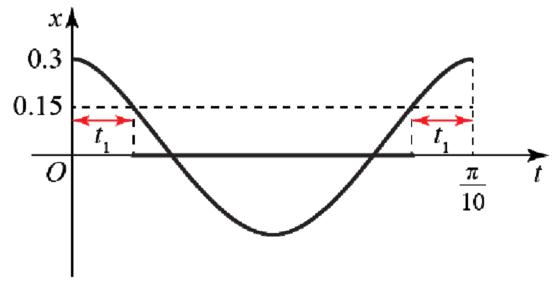
$x = a\cos\omega t$

$0.15 = 0.3\cos 20t_1 \Rightarrow \cos 20t_1 = \frac{1}{2}$

$20t_1 = \frac{\pi}{3} \Rightarrow t_1 = \frac{\pi}{60}$

The total time for which the length of the spring is less than 0.75 m is given by

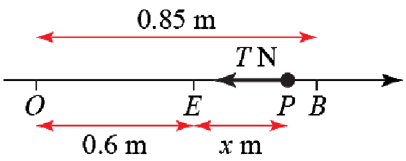
$T' = T - 2t_1 = \frac{\pi}{10} - 2 \times \frac{\pi}{60} = \frac{\pi}{15}$ s



When the length of the spring is less than 0.75 m, the extension of the spring, x m, is less than 0.15 m. This sketch shows you that if the first time where the extension is 0.15 m is t_1 s, the length of time for which the extension is less than 0.15 m is $\left(\frac{\pi}{10} - 2t_1\right)$s.

The length of time for which the length of the spring is less than 0.75 m is $\frac{\pi}{15}$s.

36 a



As you will use Newton's second law in this question, it is safer to use base SI units. So convert the distances in cm to m.

Let E be the point where $OE = 0.6$ m.

When P is at a general point in its motion, let $EP = x$ metres and the force of the spring on P be T newtons.

Hooke's Law

$T = \frac{\lambda x}{l} = \frac{12x}{0.6} = 20x$

R($\rightarrow$) $\mathbf{F} = m\mathbf{a}$

$-T = 0.8\ddot{x}$

$-20x = 0.8\ddot{x}$

$\ddot{x} = -25x = -5^2 x$

When x is positive, the tension is the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 5$. The period, T seconds, is given by

$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$ s as required.

36 b The amplitude of the motion is 0.25 m.

The maximum magnitude of the acceleration is given by

$|\ddot{x}| = |\omega^2 a| = 25 \times 0.25 = 6.25$

The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

The maximum magnitude of the acceleration is 6.25 m s^{-2}.

c $x = a\cos\omega t$

$\dot{x} = -a\omega\sin\omega t$

At $t = 2$

$\dot{x} = -0.25 \times 5\sin(5\times 2) = -1.25\sin 10$

$= +0.680\,026\ldots$

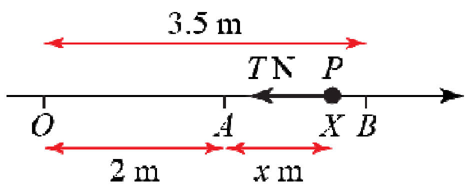
The speed of P as it passes through C is 0.68 m s^{-1} (2 s.f.).

You can drive an equation connecting velocity with time by differentiating $x = a\cos\omega t$ with respect to t. You obtain $v = \dot{x} = \frac{dx}{dt} = -a\omega\sin\omega t$. This equation is particularly useful when you are asked about the direction of motion of a particle. As the v is squared in $v^2 = \omega^2(a^2 - x^2)$, values of v found using this formula have an ambiguous $\pm$ sign.

d As the sign of $\dot{x}$ in part **c** is positive, P is travelling in the direction of x increasing as it passes through C.

As it passes through C, P is moving away from O towards B.

37 a



When P is at the point X, where $AX = x$ m, let the tension in the spring be

Hooke's law

$T = \frac{\lambda x}{l} = \frac{21.6 \times x}{2} = 10.8x$

R($\rightarrow$)$\mathbf{F} = m\mathbf{a}$

$-T = 0.3\ddot{x}$

$-10.8x = 0.3\ddot{x}$

$\ddot{x} = -36x = -6^2 x$

When x is positive, the tension in the spring is acting in the direction of x decreasing, so T has a negative sign in the equation of motion.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P is performing simple harmonic motion about A with $\omega = 6$.

The period of motion Ts is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}$ s as required.

37 b At A, $x = 0$

$$v^2 = \omega^2(a^2 - x^2) = 36(1.5^2 - 0^2) = 81$$

$$v = \sqrt{81} = 9$$

The speed of P at A is 9 m s^{-1}.

c At C, $x = \frac{1.5}{2} = 0.75$

$$x = a\cos\omega t$$

$$0.75 = 1.5\cos 6t$$

$$\cos 6t = \frac{1}{2} \Rightarrow 6t = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{18}$$

The time when P first reaches C is the smallest positive value of t for which this equation is true. In principle, in simple harmonic motion, P will reach this point infinitely many times.

P reaches C for the first time after $\frac{\pi}{18}$ s.

d Before impact, the linear momentum of P is
$m_1u = 0.3 \times 9$ N s $= 2.7$ N s

Let the velocity of the combined particle R immediately after impact be U m s^{-1}.

After impact, the linear momentum of R is
$m_2U = 0.5U$ N s

Conservation of linear momentum
$0.5U = 2.7 \Rightarrow U = 5.4$

Conservation of linear momentum is an M1 topic and is assumed, and can be tested, in any of the subsequent Mechanics modules.

For R

$$R(\rightarrow) \quad -T = 0.5\ddot{x}$$

$$-10.8x = 0.5\ddot{x}$$

$$\ddot{x} = -21.6x$$

When R is at X, Hooke's law gives $T = 10.8x$, exactly as in part **a**. There is no need to repeat the working in part **d**.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, after the impact R is performing simple harmonic motion about A with $\omega^2 = 21.6$.

$$v = U = a\omega$$

$$5.4 = a\sqrt{21.6}$$

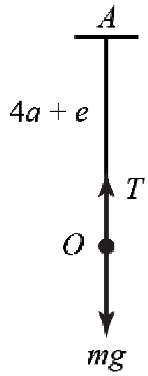
$$a = \frac{5.4}{\sqrt{21.6}} = 1.161\ldots$$

As R is performing simple harmonic motion about A, the speed of R immediately after the impact is the maximum speed of R during its motion. The maximum speed during simple harmonic motion is given by $v = a\omega$.

The amplitude of the motion is
1.16 m (3 s.f.)

No accuracy is specified in the question and the accurate answer, $\frac{3\sqrt{15}}{10}$ m, or any reasonable approximation would be accepted.

38 a



A particle attached to one end of an elastic string will oscillate about the equilibrium position. When solving problems about vertical oscillations, you often have to begin by finding the point of equilibrium. In this case, the oscillations later in the question have centre O.

At the equilibrium level, let $AO = 4a + e$, where e is the extension of the string.

Hooke's law

$$T = \frac{\lambda e}{l} = \frac{8mge}{4a} = \frac{2mge}{a}$$

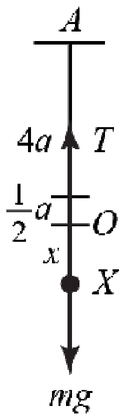
R($\uparrow$)$T = mg$

Hence

$$mg = \frac{2mge}{a} \Rightarrow e = \frac{a}{2}$$

$$AO = 4a + e = 4a + \frac{a}{2} = \frac{9a}{2}$$

38 b



When P is at a general point, X say, of its motion, let $OX = x$.

To show that P is moving with simple harmonic motion, you have to show that, at a general point in its motion, the equation of motion of P has the form $\ddot{x} = -\omega^2 x$, where ω is a positive constant.

At this point, the extension of the string is $\frac{1}{2}a + x$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{8mg\left(\frac{1}{2}a + x\right)}{4a}$$

$$= \frac{4mga + 8mgx}{4a} = mg + \frac{2mgx}{a} \quad \textbf{(1)}$$

Newton's second law

R($\downarrow$) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m\ddot{x} \quad \textbf{(2)}$$

Hooke's law and Newton's second law give you two equations from which you eliminate the tension, T.

Substituting **(1)** into **(2)**

$$mg - \left(mg + \frac{2mgx}{a}\right) = m\ddot{x}$$

$$-\frac{2mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a}x$$

Comparing this equation with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P moves with simple harmonic motion about O and

$$\omega = \sqrt{\left(\frac{2g}{a}\right)}.$$

The period of motion T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{a}{2g}\right)} = \pi\sqrt{\left(\frac{2a}{g}\right)}, \text{ as required.}$$

38 c The maximum speed is given by $v = a\omega$

$$\frac{1}{2}\sqrt{ga} = d\sqrt{\left(\frac{2g}{a}\right)}$$

As the particle is pulled down a distance *d* from the equilibrium position and released from rest, *d* is the amplitude of the motion.

$$\frac{1}{4}ga = d^2 \times \frac{2g}{a}$$

Squaring both sides of the previous line.

$$d^2 = \frac{1}{8}a^2$$

$$d = \frac{1}{2\sqrt{2}}a$$

d As $a > \frac{1}{2}a,$ the string will become slack during its motion. The subsequent motion of *P* will be partly under gravity, partly simple harmonic motion.

Challenge

1 $a = 8x\frac{\mathrm{d}x}{\mathrm{d}t}$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = 8xv$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 8x$$

$$v = \int 8x\mathrm{d}x$$

$$v = 4x^2 + c$$

When $t = 0, x = 0$ and $v = -k,$ so $c = -k$

$$v = 4x^2 - k$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x^2 - k$$

The displacement x is maximum when $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

i.e. $4x^2 - k = 0$

$$x^2 = \frac{k}{4}$$

$$x = \frac{\sqrt{k}}{2}$$

Therefore the distance of the particle from the origin never exceeds $\frac{\sqrt{k}}{2}.$

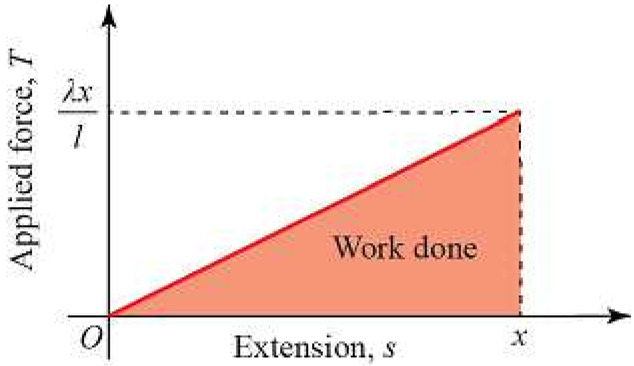
2 a Due to equivalence of work and energy:
energy stored = work done in stretching the string.

Work done in stretching the string is given by the area under the line (see graph):

$$\text{area} = \int_0^x \frac{\lambda s}{l}\, ds$$

$$\text{area} = \left[\frac{\lambda s^2}{2l}\right]_0^x$$

$$\text{area} = \frac{\lambda x^2}{2l}$$



b Work done = change in elastic potential energy stored by string

$$\text{Work done} = \frac{\lambda}{2l}\left(b^2 - a^2\right)$$

$$\text{Work done} = \frac{\lambda}{2l}(b+a)(b-a)$$

$$\text{Work done} = \frac{1}{2}\left(\frac{\lambda b}{l} + \frac{\lambda a}{l}\right)(b-a)$$

$$\text{Work done} = \tfrac{1}{2}(T_b + T_a)(b-a)$$

Work done = mean tension × distance moved as required.