Exercise 2A

1 a

$\frac{1}{2}(r(r+1)-r(r-1))$

$=\frac{1}{2}(r^2+r-r^2+r)$

$=\frac{1}{2}(2r)$

$=r$

$=\text{LHS}$

Consider RHS.

Expand and simplify.

b

$\sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n} r(r+1)-\frac{1}{2}\sum_{r=1}^{n} r(r-1)$

Use above.

$r=1 \quad \frac{1}{2}\times 1\times 2 \quad -\frac{1}{2}\times 1\times 0$

$r=2 \quad \frac{1}{2}\times 2\times 3 \quad -\frac{1}{2}\times 2\times 1$

$r=3 \quad \frac{1}{2}\times 3\times 4 \quad -\frac{1}{2}\times 3\times 2$

$\ldots \quad \ldots$

$r=n-1 \quad \frac{1}{2}(n-1)(n) \quad -\frac{1}{2}(n-1)(n-2)$

$r=n \quad \frac{1}{2}n(n+1) \quad -\frac{1}{2}n(n-1)$

Hence $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Use method of differences.

When you add, all terms cancel except $\frac{1}{2}n(n+1)$.

Use the information given and equate the summations.

2 $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\frac{1}{2r(r+1)}-\sum_{r=1}^{n}\frac{1}{2(r+1)(r+2)}$ ←

Use method of differences.

Put $r=1$ $\quad\frac{1}{2\times1\times2}-\frac{1}{2\times2\times3}$

$r=2$ $\quad\frac{1}{2\times2\times3}-\frac{1}{2\times3\times4}$

All terms cancel except first and last.

$r=3$ $\quad\frac{1}{2\times3\times4}-\frac{1}{2\times4\times5}$

⋮

$r=n$ $\quad\frac{1}{2n(n+1)}-\frac{1}{2(n+1)(n+2)}$

Adding the first and last terms we have

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

First and last from above.

$$=\frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$$

Simplify.

$$=\frac{n^2+3n+2-2}{4(n+1)(n+2)}$$

$$=\frac{n(n+3)}{4(n+1)(n+2)}$$

3 a $\frac{1}{r(r+2)}\equiv\frac{A}{r}+\frac{B}{r+2}$ ←

Set $\frac{1}{r(r+2)}$ identical to $\frac{A}{r}+\frac{B}{r+2}$.

$\equiv\frac{A(r+2)+Br}{r(r+2)}$ ←

Add the two fractions.

$1\equiv A(r+2)+Br$

Put $r = 0$

$1 = 2A$

$A = \frac{1}{2}$

Put $r = 1$

$1 = \frac{1}{2}(3)+B$

$B = -\frac{1}{2}$

$\therefore \frac{1}{r(r+2)}=\frac{1}{2r}-\frac{1}{2(r+2)}$

3 b $\sum_{r=1}^{n}\frac{1}{r(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}-\sum_{r=1}^{n}\frac{1}{2(r+2)}$

Use method of differences.

$r=1 \quad \frac{1}{2\times 1}-\frac{1}{2\times 3}$

$r=2 \quad \frac{1}{2\times 2}-\frac{1}{2\times 4}$

$r=3 \quad \frac{1}{2\times 3}-\frac{1}{2\times 5}$

$\vdots$

$r=n-1 \quad \frac{1}{2(n-1)}-\frac{1}{2(n+1)}$

$r=n \quad \frac{1}{2n}-\frac{1}{2(n+2)}$

All terms cancel except $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{2(n+1)}$ and $\frac{1}{2(n+2)}$

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{1}{2}+\frac{1}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$$

$$=\frac{2(n+1)(n+2)+(n+1)(n+2)-2(n+2)-2(n+1)}{4(n+1)(n+2)}$$

$$=\frac{2n^2+6n+4+n^2+3n+2-2n-4-2n-2}{4(n+1)(n+2)}$$

$$=\frac{3n^2+5n}{4(n+1)(n+2)}$$

$$=\frac{n(3n+5)}{4(n+1)(n+2)}$$

4 a $\frac{1}{(r+2)(r+3)}\equiv\frac{A}{r+2}+\frac{B}{r+3}$ ← Set $\frac{1}{(r+2)(r+3)}$ identical to $\frac{A}{r+2}+\frac{B}{r+3}$.

$\equiv\frac{A(r+3)+B(r+2)}{(r+2)(r+3)}$ ← Add the two fractions.

$1\equiv A(r+3)+B(r+2)$ ← Compare numerators as they are equivalent.

$r=-3 \Rightarrow B=-1$

$r=-2 \Rightarrow A=1$ ← Solve for A and B.

$\therefore \frac{1}{(r+2)(r+3)}=\frac{1}{r+2}-\frac{1}{r+3}$

4 b $\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)} \equiv \sum_{r=1}^{n}\frac{1}{(r+2)} - \sum_{r=1}^{n}\frac{1}{(r+3)}$

Use the method of differences.

$r=1 \quad \frac{1}{3} - \frac{1}{4}$

$r=2 \quad \frac{1}{4} - \frac{1}{5}$

$r=3 \quad \frac{1}{5} - \frac{1}{6}$

$\vdots$

$r=n \quad \frac{1}{n+2} - \frac{1}{n+3}$

All cancel except first and last.

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$

$$= \frac{n+3-3}{3(n+3)}$$

$$= \frac{n}{3(n+3)}$$

5 a $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$

b $\sum_{r=1}^{n}\frac{r}{(r+1)!} \equiv \sum_{r=1}^{n}\frac{1}{r!} - \sum_{r=1}^{n}\frac{1}{(r+1)!}$

Use given.

$r=1 \quad \frac{1}{1!} - \frac{1}{2!}$

Use method of differences.

$r=2 \quad \frac{1}{2!} - \frac{1}{3!}$

$r=3 \quad \frac{1}{3!} - \frac{1}{4!}$

$\vdots$

$r=n \quad \frac{1}{n!} - \frac{1}{(n+1)!}$

All cancel except first and last term.

$\therefore$

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

6 $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{1}{r^2} - \sum_{r=1}^{n} \frac{1}{(r+1)^2}$

Use given.

$r=1 \quad \frac{1}{1} - \frac{1}{2^2}$

$r=2 \quad \frac{1}{2^2} - \frac{1}{3^2}$

$r=3 \quad \frac{1}{3^2} - \frac{1}{4^2}$

$\vdots$

$r=n \quad \frac{1}{n^2} - \frac{1}{(n+1)^2}$

Use method of differences.

All terms cancel except first and last.

So adding the remaining terms we have

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$

Simplify.

7 a $\frac{1}{(2r+3)(2r+5)} = \frac{A}{2r+3} + \frac{B}{2r+5}$

$1 = A(2r+5) + B(2r+3)$

$1 = 2A = -2B$

Let $f(r) = \frac{1}{2r+3}$

$$\sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+3} - \frac{1}{2r+5}\right)$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(f(r) - f(r+1)\right) = \frac{1}{2}(f(1) - f(n+1))$$

$$= \frac{1}{2}\left(\frac{1}{2+3} - \frac{1}{2(n+1)+3}\right) = \frac{1}{2}\left(\frac{2n+5-5}{5(2n+5)}\right)$$

$$= \frac{n}{10n+25}$$

So $a = 10,\ b = 25$

7 b $n=1$: $\text{RHS}=\dfrac{1}{10+25}=\dfrac{1}{35}=\dfrac{1}{5\times 7}=\text{LHS}$

Assume true for $n=k$

$$\sum_{r=1}^{k}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}$$

$n=k+1$:

$$\sum_{r=1}^{k+1}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}+\frac{1}{(2k+5)(2k+7)}$$

$$=\frac{1}{2k+5}\left(\frac{k}{5}+\frac{1}{2k+7}\right)=\frac{1}{2k+5}\left(\frac{2k^2+7k+5}{5(2k+7)}\right)$$

$$=\frac{k+1}{10(k+1)+25}$$

so true for $n=k+1$ if true for $n=k$

true for $n=1$ so true for all natural numbers

8 $\dfrac{8}{(3r-2)(3r+4)}=\dfrac{A}{3r-2}+\dfrac{B}{3r+4}$

$8=A(3r+4)+B(3r-2)$

$8=6A=-6B$

So $A=\dfrac{4}{3}$ and $B=\dfrac{-4}{3}$

Let $\mathrm{f}(r)=\dfrac{1}{3r-2}$

$$\sum_{r=1}^{n}\frac{8}{(3r-2)(3r+4)}=\frac{4}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2}-\frac{1}{3r+4}\right)$$

$$=\frac{4}{3}\sum_{r=1}^{n}\left(\mathrm{f}(r)-\mathrm{f}(r+2)\right)$$

$$=\frac{4}{3}(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$$

$$=\frac{4}{3}\left(1+\frac{1}{4}-\frac{1}{3n+1}-\frac{1}{3n+4}\right)$$

$$=\frac{4}{3}\left(\frac{\frac{5}{4}(3n+1)(3n+4)-(3n+1)-(3n+4)}{(3n+1)(3n+4)}\right) \quad \mathbf{1}$$

$$=\frac{15n^2+25n+\frac{20}{3}-8n-\frac{20}{3}}{(3n+1)(3n+4)}=\frac{n(15n+17)}{(3n+1)(3n+4)}$$

So $a=15$ and $b=17$.

9 Let $\mathrm{f}(r) = (r-1)^2$

$$\sum_{r=1}^{n} (r+1)^2 - (r-1)^2 = \mathrm{f}(n+2) + \mathrm{f}(n+1) - \mathrm{f}(2) - \mathrm{f}(1)$$

$$= (n+1)^2 + n^2 - 1 = 2n^2 + 2n = 2n(n+1)$$

So $a = 2$

Chapter review 2

1 a $$\frac{2}{(r+2)(r+4)}=\frac{A}{r+2}+\frac{B}{r+4}$$
$$2=A(r+4)+B(r+2)$$
$$2=2A=-2B$$
$$\frac{2}{(r+2)(r+4)}=\frac{1}{r+2}-\frac{1}{r+4}$$

b Let $\mathrm{f}(r)=\frac{1}{r+2}$
$$\sum_{r=1}^{n}\frac{2}{(r+2)(r+4)}=\sum_{r=1}^{n}\left(\frac{1}{r+2}-\frac{1}{r+4}\right)$$
$$=\sum_{r=1}^{n}\big(\mathrm{f}(r)-\mathrm{f}(r+2)\big)$$
$$=\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2)$$
$$=\frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}$$
$$=\frac{7(n+3)(n+4)-12(n+3)-12(n+4)}{12(n+3)(n+4)}$$
$$=\frac{7n^2+25n}{12(n+3)(n+4)}$$

2 a $$\frac{4}{(4r-1)(4r+3)}=\frac{A}{4r-1}+\frac{B}{4r+3}$$
$$4=A(4r+3)+B(4r-1)$$
$$4=4A=-4B$$
$$\frac{4}{(4r-1)(4r+3)}=\frac{1}{4r-1}-\frac{1}{4r+3}$$

b Let $\mathrm{f}(r)=\frac{1}{4r-1}$
$$\sum_{r=1}^{n}\frac{4}{(4r-1)(4r+3)}=\sum_{r=1}^{n}\left(\frac{1}{4r-1}-\frac{1}{4r+3}\right)$$
$$=\sum_{r=1}^{n}\big(\mathrm{f}(r)-\mathrm{f}(r+1)\big)=\mathrm{f}(1)-\mathrm{f}(n+1)$$
$$=\frac{1}{3}-\frac{1}{4n+3}$$
$$=\frac{4n}{3(4n+3)}$$

2 c $\sum_{r=100}^{200} \frac{4}{(4r-1)(4r+3)}$

$= \sum_{r=1}^{200} \frac{4}{(4r-1)(4r+3)} - \sum_{r=1}^{99} \frac{4}{(4r-1)(4r+3)}$

$= \frac{800}{3(803)} - \frac{396}{3(399)} = \frac{404}{320397} = 0.00126 \text{ (3 s.f.)}$

3 a $(r+1)^3 - (r-1)^3$

$= r^3 + 3r^2 + 3r + 1 - (r^3 - 3r^2 + 3r - 1)$

$= 6r^2 + 2$

b $r^2 = \frac{1}{6}\left((r+1)^3 - (r-1)^3 - 2\right)$

$\sum_{r=1}^{n} (r+1)^3 - (r-1)^3 = (n+1)^3 + n^3 - 1$

$= 2n^3 + 3n^2 + 3n$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2\sum_{r=1}^{n} 1\right)$

$= \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right) = \frac{1}{6}n(n+1)(2n+1)$

4 $\frac{4}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$

$4 = A(r+3) + B(r+1)$

$4 = 2A = -2B$

Let $\text{f}(r) = \frac{1}{r+1}$

$\sum_{r=1}^{n} \frac{4}{(r+1)(r+3)} = 2\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$= 2(\text{f}(1) + \text{f}(2) - \text{f}(n+1) - \text{f}(n+2))$

$= 2\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = 2\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$

$= \frac{1}{3}\left(\frac{5(n+2)(n+3) - 6(n+2) - 6(n+3)}{(n+2)(n+3)}\right)$

$= \frac{5n^2 + 13n}{3(n+2)(n+3)}$

5 $\sum_{r=1}^{n}\left((r+1)^3-(r-1)^3\right)=(n+1)^3+n^3-1$

$\sum_{r=n}^{2n}(r+1)^3-(r-1)^3$

$=(2n+1)^3+(2n)^3-1-(n^3+(n-1)^3-1)$

$=16n^3+12n^2+6n-(2n^3-3n^2+3n-2)$

$=14n^3+15n^2+3n+2$

So $a=14,\ b=15,\ c=3$ and $d=2$

6 a $\frac{3}{(3r+1)(3r+4)}=\frac{A}{3r+1}+\frac{B}{3r+4}$

$3=A(3r+4)+B(3r+1)$

$3=3A=-3B$

Let $f(r)=\frac{1}{3r+1}$

$\sum_{r=1}^{n}\frac{3}{(3r+1)(3r+4)}=\sum_{r=1}^{n}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)$

$=\sum_{r=1}^{n}\left(f(r)-f(r+1)\right)=f(1)-f(n+1)$

$=\frac{1}{4}-\frac{1}{3n+4}=\frac{3n}{12n+16}$

So $a=3,\ b=12$ and $c=16$

b $\sum_{r=n}^{2n}\frac{3}{(3r+1)(3r+4)}=\frac{6n}{24n+16}-\frac{3(n-1)}{12(n-1)+16}$

$=\frac{3}{4}\left(\frac{n(3n+1)-(n-1)(3n+2)}{(3n+1)(3n+2)}\right)$

$=\frac{3}{4}\left(\frac{2n+2}{(3n+1)(3n+2)}\right)=\frac{3(n+1)}{2(3n+1)(3n+2)}$

7 $\frac{2r+1}{r(r+1)}=\frac{1}{r}+\frac{1}{r+1},f(r)=\frac{1}{r}$

$\sum_{r=1}^{n}\frac{2r+1}{r(r+1)}=f(1)+f(2)+f(2)+f(3)+\ldots+$

$f(n-1)+f(n)+f(n)+f(n+1)$

Terms don't cancel

8 $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

$1 = A(r+2) + Br$

$1 = 2A = -2B$

Let $\mathrm{f}(r) = \frac{1}{r}$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$= \frac{1}{2}(\mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2))$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4} - \frac{1}{2}\frac{2n+3}{(n+1)(n+2)}$$

$$= \frac{3}{4} - \frac{1}{2}\frac{2n+3}{(n+1)(n+2)}$$

So $a = 2$ and $b = 3$

9 a $\frac{4}{(2r+1)(2r+5)} = \frac{A}{2r+1} + \frac{B}{2r+5}$

$4 = A(2r+5) + B(2r+1)$

$4 = 4A = -4B$

So $A = 1$ and $B = -1$

$$\frac{4}{(2r+1)(2r+5)} = \frac{1}{(2r+1)} - \frac{1}{(2r+5)}$$

b Let $\mathrm{f}(r) = \frac{1}{2r+1}$

$$\sum_{r=16}^{25} \frac{4}{r(2r+5)} = \sum_{r=16}^{25}\left(\frac{1}{2r+1} - \frac{1}{2r+5}\right)$$

$= \mathrm{f}(16) - \mathrm{f}(18) + \mathrm{f}(17) - \mathrm{f}(19) + \mathrm{f}(18) - \ldots$

$+\mathrm{f}(25) - \mathrm{f}(27)$

$= \mathrm{f}(16) + \mathrm{f}(17) - \mathrm{f}(26) - \mathrm{f}(27)$

$= \frac{1}{33} + \frac{1}{35} - \frac{1}{53} - \frac{1}{55} = 0.0218$ (4 d.p.)

Challenge

a $\ln\left(1+\frac{1}{r+2}\right)=\ln\left(\frac{r+3}{r+2}\right)=\ln(r+3)-\ln(r+2)$

Let $f(r)=\ln(r+2)$

$$\sum_{r=1}^{n}\ln\left(1+\frac{1}{r+2}\right)=f(n+1)-f(1)$$

$$=\ln(n+3)-\ln 3=\ln\frac{n+3}{3}$$

$$n=30:\sum_{r=1}^{30}\ln\left(1+\frac{1}{r+2}\right)=\ln 11$$

So $k=11$

b $\frac{18}{r(r+3)}=\frac{A}{r}+\frac{B}{r+3}$

$18=A(r+3)+Br$

$18=3A=-3B$

Let $f(r)=\frac{1}{r}$

$$\sum_{r=1}^{n}\frac{18}{r(r+3)}=6\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+3}\right)$$

$$=6(f(1)+f(2)+f(3)-f(n+1)-f(n+2)-f(n+3))$$

$$=6\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$=\frac{11(n+1)(n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$-6\frac{(n+1)(n+2)+(n+1)(n+3)+(n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$=\frac{11n^3+48n^2+49n}{(n+1)(n+2)(n+3)}$$

So $a=11,\ b=48$ and $c=49$