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A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension
Insert

Practice Paper – Set 4

Time allowed: 2 hours

INFORMATION FOR CANDIDATES

- This insert contains the article for Section B.
- This document consists of **4** pages. Any blank pages are indicated.

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Fractals

The Koch snowflake

The Koch snowflake is a shape that can be constructed by an iterative process, sta[illegible] equilateral triangle shown in Fig. C1.1. In the first iteration, each side is divided ir[illegible] parts and an equilateral triangle is constructed on the middle segment to give the polygon shown in Fig. C1.2. In the second iteration, each side of the polygon in Fig. C1.2 is divided into three equal parts and an equilateral triangle is constructed on the middle segment to give the polygon shown in Fig. C1.3.

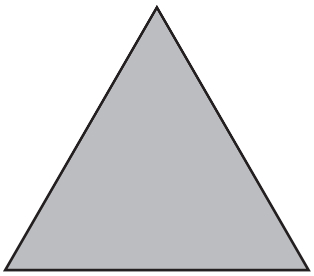

Fig. C1.1

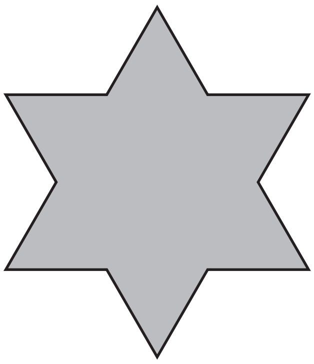

Fig. C1.2

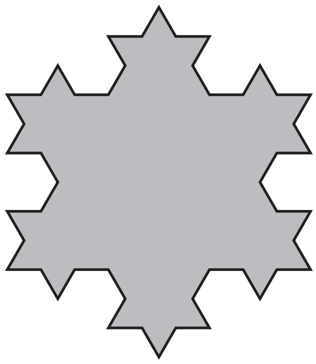

Fig. C1.3

The result of the sixth iteration is shown in Fig. C2.

The limit of this iterative process is the Koch snowflake. The snowflake encloses a region with finite area, but it does not have a finite perimeter, as the perimeter of the iterations increases without limit. This shape was first described by the Swedish mathematician Helge von Koch at the beginning of the 20th century.

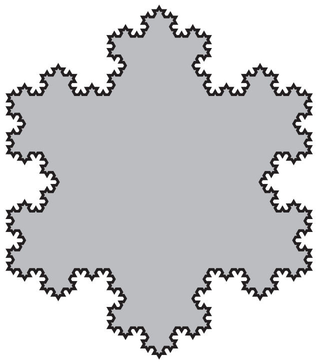

Fig. C2

Fractals

The Koch snowflake is an example of a fractal; these are geometric objects which have some self-similarity – that is, the same pattern can be seen at different scales; fractals are usually created by repeating a simple process and continuing it indefinitely.

The dragon curve

The dragon curve was first described in 1967. In this case the process starts with a horizontal line segment AB of length 2 units. In the first iteration, this is replaced by two perpendicular lines each of length $\sqrt{2}$. In the next iteration, these two lines are themselves each replaced by a pair of perpendicular lines; going from A to B, the replacement pairs are located alternately to the right and to the left of the originals, as shown by the arrows in Fig. C3. This replacement process continues in each new iteration.

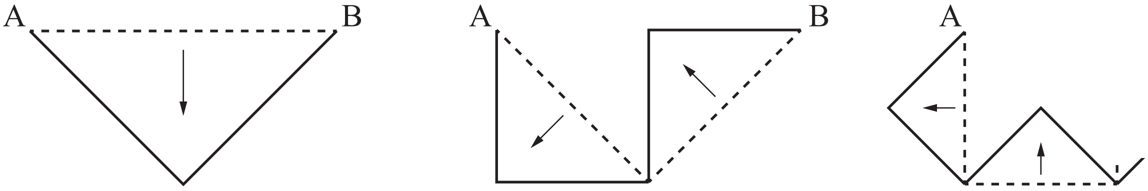

Fig. C3

Fig. C4 shows some subsequent iterations; the appearance of the curve begins to change from a series of one-dimensional line segments to something which seems to be filling up two-dimensional regions of the plane completely as smaller and smaller squares are produced.

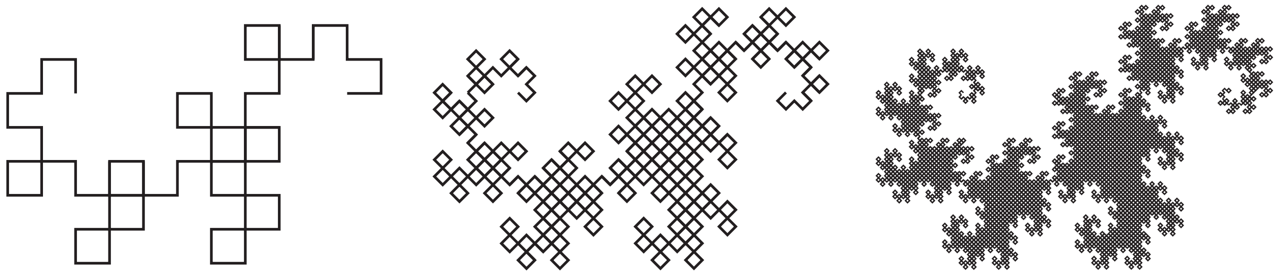

Fig. C4

Fractal dimension

Each stage of the iterative process for the Koch snowflake results in every line being replaced by 4 lines each having $\frac{1}{3}$ of the original length. For such a process, the fractal dimension, D, is defined by $D = -\frac{\ln N}{\ln s}$ where N is the factor by which the number of lines increases and s is the factor by which the line length is scaled down. So for the Koch snowflake, $N = 4$ and $s = \frac{1}{3}$ giving $D = -\frac{\ln 4}{\ln\left(\frac{1}{3}\right)} = 1.2619$.

The term 'fractal dimension' was first used by Benoit Mandelbrot in 1975. Earlier, he had considered the problem of measuring a nation's coastline. Imagine measuring the coastline of Britain with a one kilometre long measuring stick; this is equivalent to approximating the coastline by a polygon with sides of length 1 km. If the coastline was then approximated by a polygon with sides of length 500 m (equivalent to measuring with a 500 metre long measuring stick), that polygon would have more than twice as many sides because the measuring stick could follow the coastline more closely. This idea can be applied to fractal curves and leads to the definition of fractal dimension given in the previous paragraph.

Suppose the Koch snowflake is drawn by starting with an equilateral triangle of side 1 unit. If a measuring stick of length 1 unit is used to measure the final Koch snowflake then it could only follow the detail of the outline shown in Fig. C1.1 to give a total length of 3 sticks. However, a measuring stick of length $\frac{1}{3}$ unit would be able to follow the detail shown in Fig. C1.2 and give a total length of 12 sticks. Scaling down the measuring stick again would enable the detail of Fig. C1.3 to be followed, and so on.

The fractal dimension is an extension of the idea of dimension for ordinary shapes. The formula is equivalent to $N = s^{-D}$, and this form corresponds to how changing units in ordinary works.

For an ordinary one-dimensional line ($D = 1$), changing the units from cm to mm (s in the number N of units in the length increasing by a factor of 10; that is $N = s^{-1}$.

For areas of two-dimensional shapes, square units such as cm^2 or m^2 are used. Scaling the unit of length from cm to m by a factor of 100 results in the number of square units for an area changing by a factor of $\frac{1}{100^2}$. So in this case $N = s^{-2}$, that is $D = 2$.

It can be shown that the fractal dimension of the dragon curve is 2. Fractal dimension can also be applied to non-iterative shapes; the fractal dimension of the coast of Britain has been estimated as 1.25.