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A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension
Insert

Practice Paper – Set 1

Time allowed: 2 hours

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Approximating $n!$

Factorials

The number $1 \times 2 \times 3 \times 4 \times 5$ is usually written $5!$ and pronounced 'five factorial'. You evaluate $5!$ you need to perform four multiplications. Similarly, to calculate the value of $100!$ you need to perform 99 multiplications.

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The size of $n!$ increases rapidly for larger values of n . For example, $5! = 120$ is a 3-digit number, $10! = 3\,628\,800$ is a 7-digit number, while $20!$ has 19 digits. For large factorials, it is often the case that the exact value is not required; an approximate value is sufficient. This article presents one way of finding approximations to factorials.

The graph of $y = \ln x$

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Think about the area of the region under the graph $y = \ln x$ between $x = 1$ and $x = 8$; this region is shown in Fig. C1.1.

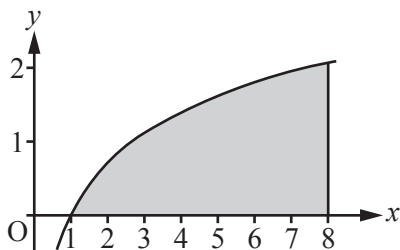


Fig. C1.1

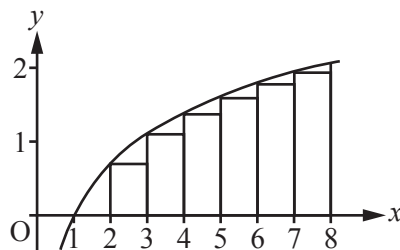


Fig. C1.2

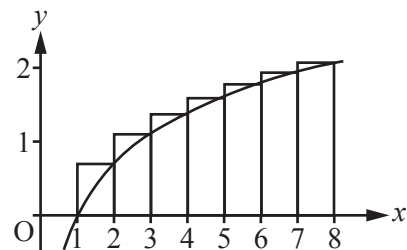


Fig. C1.3

The area of the shaded region is $\int_1^8 \ln x \, dx = 8 \ln 8 - 7$.

It is possible to find lower and upper bounds for this area by approximating the area of the region using rectangles, as shown in Fig. C1.2 and Fig. C1.3 respectively.

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The total area of the rectangles in Fig. C1.2 is $\ln 2 + \ln 3 + \dots + \ln 7$. Writing this as $\ln 7!$ and comparing it with the exact area gives $\ln 7! < 8 \ln 8 - 7$. This can be written $7! < e^{8 \ln 8 - 7}$ which simplifies to $7! < \frac{8^8}{e}$.

Finally, multiplying by 8 gives this in the more useful form: $8! < 8e\left(\frac{8}{e}\right)^8$. Similarly, comparing the total area of the rectangles in Fig. C1.3 with the exact area gives $\ln 8! > 8 \ln 8 - 7$. It follows that $8! > e^{8 \ln 8 - 7}$ so $8! > e\left(\frac{8}{e}\right)^8$.

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Combining these two results shows that $e\left(\frac{8}{e}\right)^8 < 8! < 8e\left(\frac{8}{e}\right)^8$; to an accuracy of 3 significant figures this gives $15\,300 < 8! < 122\,000$. The true value of $8!$ is 40320 so it can be seen that this method has provided bounds which are not very close to the actual value.

The same method can be used to show that $1.01 \times 10^{157} < 100! < 1.01 \times 10^{159}$. In fact, $100! \approx 9.33 \times 10^{157}$.

Better approximations

Approximating the region under the curve using trapeziums rather than rectangles provides a better approximation; this is shown in Fig. C2.

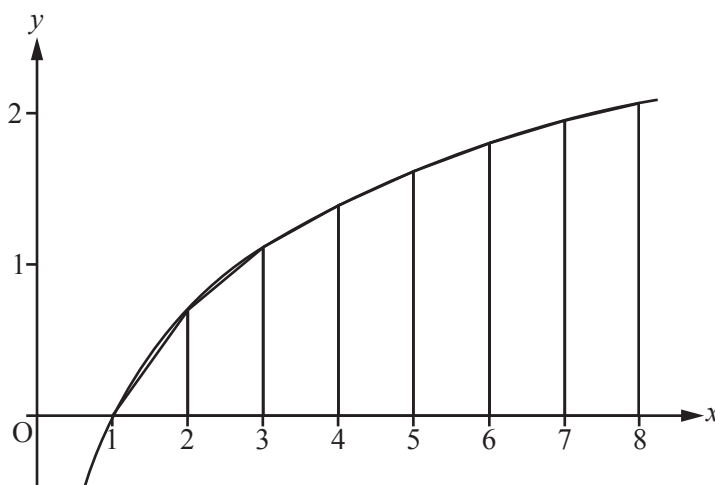


Fig. C2

The trapezium rule provides the following approximation.

$$\frac{1}{2}(\ln 1 + 2(\ln 2 + \ln 3 + \dots + \ln 7) + \ln 8) < \int_1^8 \ln x \, dx$$

Using the result $\int_1^8 \ln x \, dx = 8 \ln 8 - 7$ as given above, this simplifies to

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$$\ln 2 + \ln 3 + \dots + \ln 7 + \ln 8 < \frac{17}{2} \ln 8 - 7.$$

This can be written $\ln 8! < \frac{17}{2} \ln 8 - 7$ and this simplifies to $8! < \frac{8^{8.5}}{e^7}$ so $8! < e\sqrt{8}\left(\frac{8}{e}\right)^8$. This gives $8! < 43\,272$.

The same procedure can be used to show that $100! < e\sqrt{100}\left(\frac{100}{e}\right)^{100}$. This gives an upper bound for $100!$ of approximately 1.01×10^{158} .

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More sophisticated techniques provide better approximations. For example, Stirling's formula approximates $n!$ by the expression $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. For $n = 100$, this is accurate to within 0.1%.

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