Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Practice Paper – Set 1

Time allowed: 2 hours

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Approximating *n*!

Factorials

The number $1 \times 2 \times 3 \times 4 \times 5$ is usually written 5! and pronounced 'five factorial'. You evaluate 5! you need to perform four multiplications. Similarly, to calculate the value of 1 need to perform 99 multiplications.

The size of $n!$ increases rapidly for larger values of n. For example, $5! = 120$ is a 3-digit number, $10! = 3\,628\,800$ is a 7-digit number, while 20! has 19 digits. For large factorials, it is often the case that the exact value is not required; an approximate value is sufficient. This article presents one way of finding approximations to factorials.

The graph of $y = \ln x$

Think about the area of the region under the graph $y = \ln x$ between $x = 1$ and $x = 8$; this region is shown in Fig. C1.1.

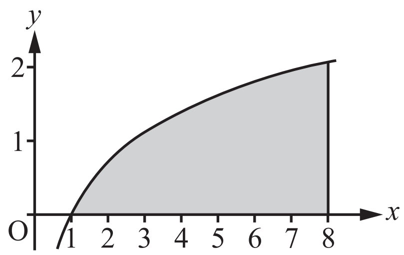

Fig. C1.1

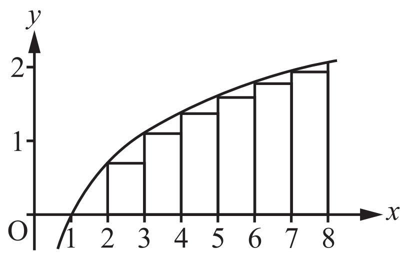

Fig. C1.2

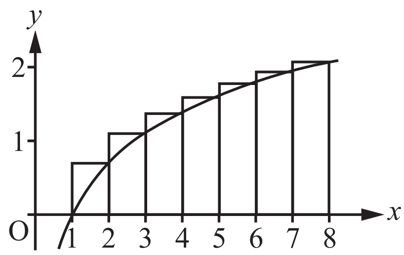

Fig. C1.3

The area of the shaded region is $\int_1^8 \ln x \, dx = 8\ln 8 - 7$.

It is possible to find lower and upper bounds for this area by approximating the area of the region using rectangles, as shown in Fig. C1.2 and Fig. C1.3 respectively.

The total area of the rectangles in Fig. C1.2 is $\ln 2 + \ln 3 + \ldots + \ln 7$. Writing this as $\ln 7!$ and comparing it with the exact area gives $\ln 7! < 8\ln 8 - 7$. This can be written $7! < e^{8\ln 8 - 7}$ which simplifies to $7! < \frac{8^8}{e^7}$.

Finally, multiplying by 8 gives this in the more useful form: $8! < 8e\left(\frac{8}{e}\right)^8$. Similarly, comparing the total area of the rectangles in Fig. C1.3 with the exact area gives $\ln 8! > 8\ln 8 - 7$. It follows that $8! > e^{8\ln 8 - 7}$ so $8! > e\left(\frac{8}{e}\right)^8$.

Combining these two results shows that $e\left(\frac{8}{e}\right)^8 < 8! < 8e\left(\frac{8}{e}\right)^8$; to an accuracy of 3 significant figures this gives $15\,300 < 8! < 122\,000$. The true value of 8! is 40 320 so it can be seen that this method has provided bounds which are not very close to the actual value.

The same method can be used to show that $1.01 \times 10^{157} < 100! < 1.01 \times 10^{159}$. In fact, $100! \approx 9.33 \times 10^{157}$.

Better approximations

Approximating the region under the curve using trapeziums rather than rectangles pr approximation; this is shown in Fig. C2.

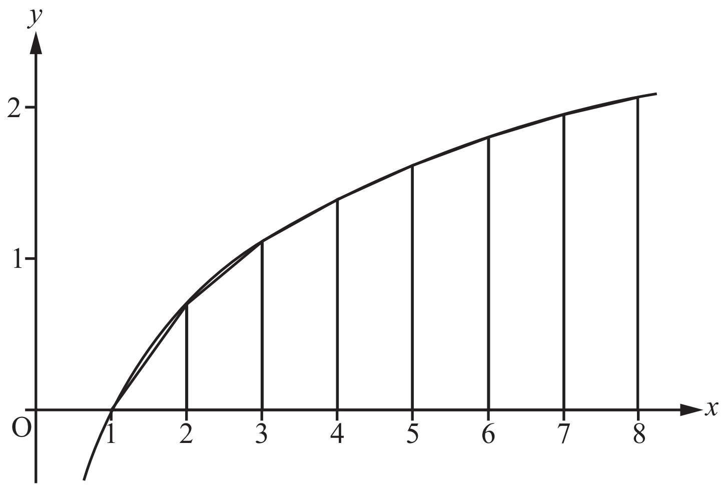

Fig. C2

The trapezium rule provides the following approximation.

$$\frac{1}{2}\left(\ln 1 + 2(\ln 2 + \ln 3 + ... + \ln 7) + \ln 8\right) < \int_1^8 \ln x \, dx$$

Using the result $\int_1^8 \ln x \, dx = 8\ln 8 - 7$ as given above, this simplifies to

$$\ln 2 + \ln 3 + ... + \ln 7 + \ln 8 < \frac{17}{2}\ln 8 - 7.$$

This can be written $\ln 8! < \frac{17}{2}\ln 8 - 7$ and this simplifies to $8! < \frac{8^{8.5}}{e^7}$ so $8! < e\sqrt{8}\left(\frac{8}{e}\right)^8$. This gives $8! < 43\,272$.

The same procedure can be used to show that $100! < e\sqrt{100}\left(\frac{100}{e}\right)^{100}$. This gives an upper bound for 100! of approximately 1.01×10^{158}.

More sophisticated techniques provide better approximations. For example, Stirling's formula approximates $n!$ by the expression $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. For $n = 100$, this is accurate to within 0.1%.