GCE A level Further Mathematics (9FM0) – Shadow Paper (Set 1)
9FM0-4A AL Further Pure 2

October 2021 Shadow Paper mark scheme

Please note that this mark scheme is not the one used by examiners for making scripts. It is intended more as a guide, indicating where marks are given for correct answers. As such, it may not show follow-through marks (marks that are awarded despite errors being made) or special cases.

It should also be noted that for many questions, there may be alternative methods of finding correct solutions that are not shown here – they will be covered in the formal mark scheme from the original paper.

This document is intended for guidance only and may differ significantly from the examiners' final mark scheme for the original paper which was published in December 2021.

Guidance on the use of codes within this document
M1 – method mark. This mark is generally given for an appropriate method in the context of the question. This mark is given for showing your working and may be awarded even if working is incorrect. A1 – accuracy mark. This mark is generally given for a correct answer following correct working. B1 – working mark. This mark is usually given when working and the answer cannot easily be separated. Some questions require all working to be shown; in such questions, no marks will be given for an answer with no working (even if it is a correct answer).

Question	Scheme	Marks	AOs
1	Identifies either 3 or 11 as a prime divisor of 165 and proceeds to apply the divisibility test for this prime number.	**M1**	3.1a

	Either $1 - 2 + 8 - 9 + 7 - 0 + 6 - 0 = 1 \times 11$ hence n is divisible by 11 Or $1 + 2 + 8 + 9 + 7 + 0 + 6 + 0 = 33 = 3 \times 11$ hence n is divisible by 3	**A1**	2.2a
	Both $1 - 2 + 8 - 9 + 7 - 0 + 6 - 0 = 1 \times 11$ hence n is divisible by 11 And $1 + 2 + 8 + 9 + 7 + 0 + 6 + 0 = 33 = 3 \times 11$ hence n is divisible by 3	**A1**	2.2a
	As also n ends in 0, it is divisible by 5, and hence as divisible by 3, 5 and 11, is divisible by $3 \times 5 \times 11 = 165$	**A1**	2.4
		(4)	
			(4 marks)

Notes:

M1: Identifies either 3 or 11 as a prime factor of 165 and proceeds to check divisibility for it.
A1: Correct method and deduction for either divisibility by 3 or by 11
A1: Correct method and deduction for both divisibility by 3 and by 11
A1: Notes also divisibility by 5 and explains why divisibility by 165 follows. The explanation may have been given in a preamble "165 = 3×5×11 so divisible by 165 if divisible by 3, 5 and 11"
The must be a correct reason for divisibility by 5, ie "last digit is 0".

Question	Scheme	Marks	AOs

2(a)	For $m, n \in \mathbb{R}_0^+$, we have $m^2, n^2 \in \mathbb{R}_0^+$ and hence $m^2 - n^2 \in \mathbb{R}_0^+$ (difference of two real numbers is a real number). $\lvert m^2 - n^2 \rvert > 0$ and so $\sqrt{\lvert m^2 - n^2 \rvert} \in \mathbb{R}_0^+$ is well defined, hence closed under A.		**B1**	2.4
			(1)	
(b)	For $m \in \mathbb{R}_0^+$, $0 \star m = \sqrt{\lvert 0^2 - m^2 \rvert} = \sqrt{m^2} = m$	Checks either side	**M1**	1.1b
	and $m \star 0 = \sqrt{\lvert m^2 - 0^2 \rvert} = \sqrt{m^2} = m$ Hence 0 is an identity*.	Checks both sides and makes conclusion.	**A1***	2.1
			(2)	
(c)	For $m \in \mathbb{R}_0^+$, we need $\sqrt{\lvert m^2 - n^2 \rvert} = 0 \Rightarrow n = \cdots$ or shows $\sqrt{\lvert m^2 - m^2 \rvert} = \sqrt{0} = 0$		**M1**	2.2a
	As $\sqrt{\lvert m^2 - m^2 \rvert} = 0$ for all $m \in \mathbb{R}_0^+$, each m is self-inverse.		**A1**	2.1
			(2)	
(d)	Checks associativity – ie evaluates m Â(n Âp) and (m Ân) Âp with letter or numbers.		**M1**	1.2
	E.g, $1 \star (2 \star 3) = 1 \star \sqrt{\lvert 2^2 - 3^2 \rvert} = 1 \star \sqrt{5} = \sqrt{\lvert 1^2 - 5 \rvert} = 2$ but $(1 \star 2) \star 3 = \sqrt{\lvert 1^2 - 2^2 \rvert} \star 3 = \sqrt{3} \star 3 = \sqrt{\lvert 3 - 3^2 \rvert} = \sqrt{6}$		**M1**	3.1a
	1 Â(2 Â3) $\neq$ (1 Â2) Â3 hence not associative so not a group.		**A1**	2.4
			(3)	
				(8 marks)

Notes:

(a)

B1: Checks difference of two non-negative real numbers is a real number and hence its modulus is a non-negative real number and therefore its (positive) squareroot is a non-negative real number, concluding closure. “Always positive” as a conclusion is B0 without consideration of the equal zero case.

NB: We follow the convention that $\sqrt{x}$ always denotes the positive squareroot of x.

(b)

M1: Checks that 0 is a left or a right identity.

A1*: Checks 0 works both sides as an identity and makes conclusion it is an identity.

(c)

M1: Realises m must be its own inverse for each m – accept if just stated m is self-inverse with no proof, or if an attempt is made to show it is self-inverse, or for an attempt to solve $\sqrt{\lvert m^2 - n^2 \rvert} = 0$

A1: Each element is self-inverse with a **full** proof given.

(d)

M1: Realises associativity must be checked in some way – may be by producing a counter example, or by attempting to evaluate both sides of the associativity axiom for a general case. A statement of the correct identity is sufficient for the mark to be awarded.

M1: Produces a suitable counter example and evaluates both sides of associativity equation. Attempts at algebraic proofs are unlikely to succeed but allow the method for e.g consideration of $m > n > p$ giving $\left||m^2 - n^2| - p^2\right| = |m^2 - n^2 - p^2|$ and $\left|m^2 - |n^2 - p^2|\right| = |m^2 - n^2 + p^2|$ but must have a correct reason to disambiguate the inner moduli. If in doubt use review.

A1: Must have provided a counter example. Deduces associativity does not hold and concludes $\mathbb{R}_0^+$ is not a group under A

Question	Scheme	Marks	AOs
3(a)	$475 = 88 \times 5 + 35$ $88 = 35 \times 2 + 18$	**M1**	1.1b
	$35 = 18 \times 1 + 17$ $18 = 17 \times 1 + 1$	**M1** **A1**	1.1b 1.1b
	$1 = 18 \times 1 - 17$ $= 18 - (35 - 18) = 18 \times 2 - 35$ $= (88 - 35 \times 2) \times 2 - 35 = 88 \times 2 - 35 \times 5$	**M1**	2.1
	$1 = 88 \times 2 - (475 - 88 \times 5) \times 5 = -5 \times 475 + 88 \times 27$ (So $a = -5$ and $b = 27$)	**A1**	1.1b
		(5)	
(b)	From (a) $88 \times 27 \equiv 1 \pmod{475}$ so multiplicative inverse of 88 is 27.	**B1ft**	2.2a
		(1)	
(c)	$x \equiv 27 \times 25 \pmod{475}$	**M1**	1.1b
	$x \equiv 675 \equiv 200 \pmod{475}$	**A1**	1.1b
		(2)	
			(8 marks)

Notes:

(a)
M1: Begins the process of applying the Euclidean algorithm, with attempt at the first two steps. Allow slips.
M1: Completes the process to the stage shown – if errors have been made at least three steps should have been made in reaching their final line (ending +1) to score this mark.
A1: Algorithm correctly carried out – as shown.
M1: Starts the process of back substitution – at least two substitutions made.
A1: Completes the process and finds the correct values for *a* and *b*.

(b)
B1ft: Deduces correct multiplicative inverse. Accept 27 or anything congruent to 27 modulo 475 or follow through their *b*.

(c)
M1: Multiplies 27 by their multiplicative inverse or any other full method to proceed to the solution, e.g. multiplying the identity found in (a) through by 27 and reducing modulo 475.
A1: $x \equiv 200 \pmod{475}$. Accept 200 or anything congruent to 200 modulo 475 as long as it is part of a correct modulo statement. (Do not accept just 675 on its own.)

Question	Scheme	Marks	AOs
4(i)	(Order of a subgroup must divide the order of a group by Lagrange's Theorem), so need to check if 13 (and/or 55) divides $61^{61} + 62^{62}$ and by FLT, e.g. $a^{13-1} = a^{12} \equiv 1$ (mod 13), so	**M1**	1.1b
	$61^{61} + 62^{2} \equiv 9^{5\times12+1} + 10^{5\times12+2} \equiv 9 + 10^{2}$ $\equiv 109 \equiv 5$ (mod 13)	**M1**	3.1a
	Hence 13 is not a divisor of $61^{61} + 62^{62}$ so not a possible order for a subgroup.	**A1**	2.2a
(ii)	$55 = 5 \times 11$ so need to check for factors of 5 and 11, using $a^{4} \equiv 1$(mod 4) and $a^{10} \equiv 1$(mod 11)	**M1**	3.1a
	$61^{61} + 62^{62} \equiv 1^{15\times4+1} + 2^{15\times4+2} \equiv 1 + 2^{2} \equiv 5 \equiv 0$ (mod 5)	**M1**	1.1b
	$61^{61} + 62^{62} \equiv 6^{10\times6+1} + 7^{10\times6+2} \equiv 6^{1} + 7^{2} \equiv 55 \equiv 0$ (mod 11)	**M1**	2.1
	As $61^{61} + 62^{62}$ divisible by both 5 and 11 it is divisible by 55 and hence this is a possible order for a subgroup.	**A1**	2.4
		(7)	

(7 marks)

Notes:

(i)

M1: For an attempt to apply a correct Fermat's Little theorem at least once in the question with either $p = 13$, p = 11 or $p = 5$ on either the 61^{61} or 62^{62} term.

M1: Applies FLT and congruence arithmetic fully to find the residue of $61^{61} + 62^{62}$ modulo 13. There will be lots of different routes, so look for an attempt to apply FLT that leads to determining if 13 is a divisor or not.

A1: $\mathbf{61^{61} + 62^{62} \equiv 5}$ **(*mod* 13)** (accept equivalents as long as it is clear it is not congruent to 0) and deduces it is not a possible order for a subgroup.

(ii)

M1: Applies checks for both 11 and 5 as divisors of $61^{61} + 62^{62}$ via similar strategy.

M1: Applies FLT with $p = 5$ to find a smaller residue modulo 5. Other routes are possible.

M1: Applies FLT with $p = 11$ to find a smaller residue modulo 11. Other routes are possible.

A1: Shows $61^{61} + 62^{62}$ congruent to 0 modulo 5 and modulo 11, and deduces 55 divides $61^{61} + 62^{62}$ hence it is a possible order for a subgroup.

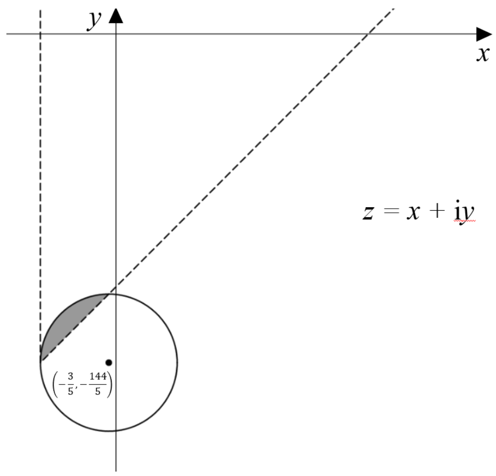
Alt:

M1: Reduces the bases modulo 55 and applies a power reduction technique using congruences for at least one of the power of 61 or 62

M1: Reduces fully by congruence arithmetic either the 61^{61} or 62^{62} term.

M1: Reduces fully by congruence arithmetic both the 61^{61} and 62^{62} terms

A1: Shows $61^{61} + 62^{62}$ congruent to 0 modulo 55, and deduces 55 divides $61^{61} + 62^{62}$ hence it is a possible order for a subgroup.

Question	Scheme	Marks	AOs
5(a)	$z = x + \mathrm{i}y \Rightarrow \lvert x - 21 + \mathrm{i}y \rvert = 6\lvert x + (y + 28)\mathrm{i} \rvert$	**M1**	1.1b
	$\Rightarrow (x - 21)^2 + y^2 = 36(x^2 + (y + 28)^2)$	**M1** **A1**	1.1b 1.1b
	$\Rightarrow 35x^2 + 35y^2 + 42x + 2016y = 21^2 - 36 \times 28^2$ $\Rightarrow x^2 + y^2 + \frac{6}{5}x + \frac{288}{5}y = -\frac{3969}{5}$		
	$\Rightarrow \left(x + \frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2 + \left(y + \frac{144}{5}\right)^2 - \left(\frac{144}{5}\right)^2 = -\frac{3969}{5}$ $\left(\Rightarrow \left(x + \frac{3}{5}\right)^2 + \left(y + \frac{144}{5}\right)^2 = 36\right)$	**M1**	2.1
	centre $-\frac{3}{5} - \frac{144}{5}\mathrm{i}$ **or** radius 6	**A1**	2.2a
	centre $-\frac{3}{5} - \frac{144}{5}\mathrm{i}$ **and** radius 6	**A1**	2.2a
		(6)	
(b)	 Circle, with centre in correct quadrant for their answer to (a)	**M1**	1.1b
	Pair of rays at roughly 45° to horizontal, with source in third quadrant OR on the circle.	**M1**	1.1b
	Correct circle and rays, circle with centre in third quadrant and spanning only quadrant 3 and 4, a ray at roughly 45° to horizontal and a ray vertically upwards, meeting at the left-most point of the circle	**A1**	3.1a
	Region between rays and inside circle shaded	**B1ft**	3.1a
		(4)	
			(10 marks)

Notes:

(a)
M1: Applies $z = x + \mathrm{i}y$ to the given equation. Use of other letters, eg $z = u + \mathrm{i}v$ is fine.

M1: Squares and uses modulus to achieve $(x+a)^2 + y^2 = K\left(x^2 + (y+b)^2\right)$

A1: Correct equation, need not be expanded. Award when first seen.
M1: Expands, gathers terms and completes the square.
A1: Either centre or radius correct. Accept coordinates for centre.
A1: Correct centre and radius. Accept coordinates for centre.

(b)

M1: Sketches their circle on an Argand diagram. Look for the centre being in the correct quadrant for their answer to (a).

M1: Pair of rays added to the sketch, at angles $\frac{\pi}{4}$ above horizontal and vertically up with vertex in the third quadrant OR somewhere on the circle. Need not stem from left-most point of circle for this mark if it stems from the third quadrant, but if not in the third quadrant it must stem from the circle.

A1: Circle (or arc) in correct position, centre in third quadrant that would span quadrants 3 and 4, with a ray at roughly 45° above horizontal and a ray vertically upwards, meeting at the left-most point of the circle.

B1ft: Area inside the circle and between the rays (minor segment) shaded provided the rays span approximately a 45° sector.

NB Only the region is asked for, so allow the marks above if only the relevant part of the circle is shown.

Question	Scheme	Marks	AOs
6	$n = 1: u_1 = (-2)^1 \times 2! = -4$ $n = 2: u_2 = (-2)^2 \times 3! = 4 \times 6 = 24$ Hence true for $n = 1$ and $n = 2$	**B1**	2.2a
	Assume true for some $n = k$ and $n = k + 1$, so $u_k = (-2)^k(k+1)!$ And $u_{k+1} = (-2)^{k+1}(k+2)!$	**M1**	2.4
	Then $u_{k+2} = 12(k+2)((-2)^k(k+1)!) - 2k((-2)^{k+1}(k+2)!)$	**M1**	1.1b
	$= (-2)^k(k+2)!\,(12 - 2k(-2))$	**M1**	1.1b
	$= (-2)^k(k+2)!\,(12+4k) = (-2)^k \times (-2)^2(k+2)(k+3)$ $= (-2)^{k+2}(k+3)!$	**A1**	2.1
	Hence if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$. As also true for $n = 1$ and $n = 2$, then true for all $n \in \bullet$ by mathematical induction.	**A1**	2.4
		(6)	
		(6 marks)	

B1: Checks the closed form works for $n = 1$ and $n = 2$

M1: Makes the inductive assumption. May use e.g. $n = k - 2$ and $n = k - 1$ instead and show true for $n = k$. It must be clear it is the closed forms they are assuming, not a recurrence form.

M1: Substitutes expression for $n = k$ and $n = k + 1$ (or equivalents) into the recurrence formula.

M1: Takes out common factors of at least $(-2)^k(k+2)!$ in their expression, or equivalent for their assumed true values. Treatment of the (-2) must be correct, but condone invisible brackets if recovered.

A1: Simplifies correctly to the required form for their assumed true values.

A1: Correct conclusion made. Depends on all three M's and the A being gained. Must convey the ideas of 1) true for $n = 1$ and $n = 2$, 2) if true for two successive cases, it is also true for the next case and 3) a suitable conclusion that it is true for all positive n.

Question	Scheme	Marks	AOs
7(a)	$I_n = \int t^{n-2} \times t^2\sqrt{1+2t^3}\,\mathrm{d}t$ $= t^{n-2} \times K(1+2t^3)^{\frac{3}{2}} - \int (n-2)t^{n-3} \times K(1+2t^3)^{\frac{3}{2}}\,\mathrm{d}t$	M1	3.1a
	$I_n = t^{n-2} \times \frac{2}{3\times 6}(1+2t^3)^{\frac{3}{2}} - \int (n-2)t^{n-3} \times \frac{2}{3\times 6}(1+2t^3)\mathrm{d}t$	A1	1.1b
	$= t^{n-2} \times \frac{1}{9}(1+2t^3)^{\frac{3}{2}} - \frac{n-2}{9}\int t^{n-3}(1+2t^3)^{\frac{1}{2}} \times (1+2t^3)\,\mathrm{d}t$ $= \frac{t^{n-2}}{9}(1+2t^3)^{\frac{3}{2}} - \frac{n-2}{9}\int t^{n-3}(1+2t^3)^{\frac{1}{2}}\,\mathrm{d}t - \frac{2(n-2)}{9}\int t^n(1+2t^3)^{\frac{1}{2}}\,\mathrm{d}t$	M1	3.1a
	$\Rightarrow 9I_n = t^{n-2}(1+2t^3)^{\frac{3}{2}} - (n-2)I_{n-3} - 2(n-2)I_n \Rightarrow I_n = \cdots$	M1	1.1b
	$I_n = \frac{t^{n-2}}{2n+5}(1+2t^3)^{\frac{3}{2}} - \frac{n-2}{2n+5}I_{n-3}$ *	A1*	2.1
		(5)	
(b)	Surface area $= 2\pi\int_0^1 y\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$	B1	1.1a
	$\frac{\mathrm{d}x}{\mathrm{d}t} = t^{\frac{17}{4}}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{2}t^{\frac{23}{4}}$	B1	1.1b
	$\int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int \frac{4\sqrt{2}}{27}t^{\frac{27}{4}}\sqrt{\left(t^{\frac{17}{4}}\right)^2 + \left(\sqrt{2}t^{\frac{23}{4}}\right)^2}\,\mathrm{d}t$	M1	1.1b
	$= \frac{4\sqrt{2}}{27}\int t^{\frac{27}{4}}\sqrt{t^{\frac{17}{2}} + 2t^{\frac{23}{2}}}\,\mathrm{d}t = \frac{4\sqrt{2}}{27}\int t^{\frac{27}{4}} \times t^{\frac{17}{4}}\sqrt{1+2t^3}\,\mathrm{d}t$ $= \frac{4\sqrt{2}}{27} = \int t^{11}\sqrt{1+2t^3}\,\mathrm{d}t$	M1	2.1
	Hence surface area $= \frac{8\pi\sqrt{2}}{27}\int t^{11}\sqrt{1+2t^3}\,\mathrm{d}t$ *	A1*	1.1b
		(5)	

Question	Scheme	Marks	AOs
(c)	$[I_2]_0^1 = \left[\frac{1}{9}(1+2t^3)^{\frac{3}{2}}\right]_0^1 = \frac{\sqrt{3}}{3} - \frac{1}{9}$ $(= 0.466 \ldots)$	B1	2.2a
	$\int_0^1 t^{11}\sqrt{1+2t^3}\,\mathrm{d}t = \left[\frac{t^9}{27}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{9}{27}[I_8]_0^1$	M1	1.1b
	$= \frac{\sqrt{3}}{9} - \frac{1}{3}\left(\left[\frac{t^6}{21}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{6}{21}[I_5]_0^1\right)$ $= \frac{\sqrt{3}}{9} - \frac{1}{3}\left(\frac{\sqrt{3}}{7} - \frac{2}{7}\left(\left[\frac{t^3}{15}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{3}{15}I_2\right)\right)$	M1	3.1a
	Total surface area is $\left(\frac{8\pi\sqrt{2}}{27}\right)\left[\frac{\sqrt{3}}{9} - \frac{1}{3}\left(\frac{\sqrt{3}}{7} - \frac{2}{7}\left(\frac{\sqrt{3}}{5} - \frac{1}{5}\left(\frac{\sqrt{3}}{3} - \frac{1}{9}\right)\right)\right)\right] = \cdots$	M1	2.1
	=awrt 0.177 (3sf) $\left(= \frac{8\pi\sqrt{2}}{27}\left(\frac{72\sqrt{3}+2}{945}\right)\right)$	A1	1.1b
		(5)	
	For the three method marks if the process is worked the other way: $[I_5]_0^1 = \left[\frac{t^3}{15}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{3}{15}[I_2]_0^1$ $\left(= \frac{\sqrt{3}}{5} - \frac{1}{5}\left(\frac{\sqrt{3}}{3} - \frac{1}{9}\right) = \frac{1+6\sqrt{3}}{45} = 0.2531 \ldots\right)$	M1	1.1b
	$[I_8]_0^1 = \left[\frac{t^6}{21}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{6}{21}[I_5]_0^2$ $\left(= \frac{\sqrt{3}}{7} - \frac{2}{7} \times \frac{1+6\sqrt{3}}{45} = \frac{33\sqrt{3}-2}{315} = 0.1751 \ldots\right)$ $[I_{11}]_0^1 = \left[\frac{t^9}{27}(1+2t^3)^{\frac{3}{2}}\right]_0^1 - \frac{9}{27}[I_8]_0^1 = \cdots$	M1	3.1a
	$\frac{\sqrt{3}}{9} - \frac{1}{3} \times \frac{33\sqrt{3}-2}{315}\left(= \frac{72\sqrt{3}+2}{945} = 0.134\right) = \cdots$	M1	2.1
	Surface area = awrt 0.177	A1	1.1b
			(15 marks)

Notes:
(a) **M1:** Splits the integrand correctly and applies integration by parts in the correct direction to achieve a form as shown in the scheme. **A1:** Correct result of applying parts, need not be simplified. **M1:** Splits the integrand to identify I_n and I_{n-3} (or allow if I_{n-1} or I_{n-2} appears due to error for this mark) in the equation. **M1:** Rearranges to make I_n the subject from an equation in I_n and I_{n-3} **A1*:** Correct completion to the given result.
(b) **B1:** Correct parametric formula for surface area given. Must include the 2π and limits, but these may be added at a later stage. The 2π must be seen in calculation for obtaining $\frac{8\pi\sqrt{2}}{27}$. **B1:** Correct derivatives of x and y with respect to t seen or implied. **M1:** Applies their derivatives and y to $\int y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$. May have included the limits and 2π here, but they are not needed for this mark. **M1:** Squares the derivatives and takes a common factor $t^{\frac{17}{2}}$ from the square root to reach appropriate form for the integral. Limits and 2π not needed for this mark. **A1*:** Reaches correct answer with no errors seen, limits included (but do not need to be justified) and the dt must be present and the 2π must have been seen and correctly processed. **(c)** **B1:** Correct value for I_2 between the limits - need not be simplified and may be seen later in the working. **M1:** Applies the reduction formula from (a) in attempt to solve the integral. This may be from I_{11} to I_8 or from I_2 to I_5 depending on the direction they are going. Allow for any application relevant to the integral (e.g between appropriate values of n which are 3 apart). **M1:** Applies the reduction formula two more times to link I_2 and I_{11}. May have evaluated at each stage or find expression before substituting limits but look for the complete process to link the two intervals. **dM1:** Applies the limits to their integral in a complete process to reach an answer. Allow if substitution happens throughout the process of reduction or at the end but it must be a complete process to find reach a value, though allow if the $\frac{8\pi\sqrt{2}}{27}$ is not included. **A1:** Must have scored all three method marks. Correct answer, awrt 0.177.

Question	Scheme	Marks	AOs
8(a)(i)	$\begin{pmatrix} -14 & 6 & -3 \\ 0 & 2 & 2 \\ p & -22 & 17 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ p-56 \end{pmatrix} = -2 \times \begin{pmatrix} 1 \\ 1 \\ 28 - \frac{1}{2}p \end{pmatrix}$		
	Corresponding eigenvalue is −2	**B1**	1.1b
		(1)	
(a)(ii)	$28 - \frac{1}{2}p = -2 \Rightarrow p = \cdots$	**M1**	1.1b
	$p = 60$ *	**A1***	1.1b
		(2)	
(a)(iii)	$\det\begin{pmatrix} -14-\lambda & 6 & -3 \\ 0 & 2-\lambda & 2 \\ 60 & -22 & 17-\lambda \end{pmatrix} = 0$ $\Rightarrow (-14-\lambda)\big((2-\lambda)(17-\lambda)+44\big) - 6(-120) - 3(-60(2-\lambda)) = 0$	**M1**	1.1b
	$\Rightarrow \lambda^3 - 5\lambda^2 - 8\lambda + 12 = 0$	**A1**	1.1b
	$(\Rightarrow (\lambda+2)(\lambda^2 - 7\lambda + 6) = 0 \Rightarrow (\lambda+2)(\lambda-6)(\lambda-1) = 0)$ Eigenvalues are (−2), 1 and 6	**A1**	1.1b
	Either $\left.\begin{matrix} -20x + 6y - 3z = 0 \\ -4y + 2z = 0 \\ 60x - 22y + 11 = 0 \end{matrix}\right\}$ or $\left.\begin{matrix} -15x + 6y - 3z = 0 \\ y + 2z = 0 \\ 60x - 22y + 16z = 0 \end{matrix}\right\} \Rightarrow x / y / z = \ldots$	**M1**	2.1
	Either $k\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda = 1$) or $m\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ $(\lambda = 6)$	**A1**	1.1b
	Both $\left.\begin{matrix} -20x + 6y - 3z = 0 \\ -4y + 2z = 0 \\ 60x - 22y + 11 = 0 \end{matrix}\right\}$ or $\left.\begin{matrix} -15x + 6y - 3z = 0 \\ y + 2z = 0 \\ 60x - 22y + 16z = 0 \end{matrix}\right\} \Rightarrow x / y / z = \ldots$	**M1**	2.1
	Both $k\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda = 1$) or $m\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ $(\lambda = 6)$	**A1**	1.1b
		(7)	
(b)	E.g. $\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & -2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	**B1ft**	2.2a
		(1)	
(c)(i)	$\dot{u} = ku \Rightarrow \int \frac{1}{u}\,\mathrm{d}u = k\int \mathrm{d}t \Rightarrow \ln u = kt(+c)$	**M1**	1.1b
	So $u = A\mathrm{e}^{kt}$ or $u = \mathrm{e}^{kt+c}$	**A1**	1.1b
		(2)	

Question	Scheme	Marks	AOs
(ii)	$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \mathbf{PDP}^{-1}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} \Rightarrow \begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \mathbf{D}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}u\\ 6v\\ -2w\end{pmatrix}$	M1	3.1b
	$\Rightarrow \begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}Ae^{t}\\ Be^{6t}\\ Ce^{-2t}\end{pmatrix}$	M1	2.2a
	$\Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \mathbf{P}\begin{pmatrix}Ae^{t}\\ Be^{6t}\\ Ce^{-2t}\end{pmatrix} = \cdots$	M1	3.4
	$\Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}Ae^{t} + Ce^{-2t}\\ 2Ae^{t} + Be^{6t} - Ce^{-2t}\\ -Ae^{t} + 2Be^{6t} - 2Ce^{-2t}\end{pmatrix}$	A1	1.1b
		(4)	
			(17 marks)

Notes:

(a)(i)
B1: For the correct eigenvalue of -2
(ii)
M1: Correct equation with their eigenvalue set up – need only see bottom equation for this.
A1*: Correct proof (full matrix calculation not necessary).
(iii)
M1: Applies $\det(\mathbf{A}-\lambda\mathbf{I})=0$ to achieve a cubic in λ (or other variable, simplification not required). Allow with p used instead of 60, and look for two correct "terms" in the expansion leading to a cubic as evidence of the expansion.
A1: Correct simplified cubic. Note this may be implied by correct answers from a calculator following a correct expansion seen for the M.
A1: Correct eigenvalues
M1: Forms and solves eigenvector equations for at least one (other than -2) eigenvalue.
A1: One correct (other) eigenvector
M1: Both eigenvectors attempted.
A1: Both (other) eigenvectors correct.

(b)
B1ft: A correct corresponding **P** and **D**, follow through on their answer to (a). Columns may be in different order, but should be consistent for their **P** and **D**.

(c)(i)
M1: Separates variables and attempts the integration (constant not required).
A1: Correct answer for $u =\ldots$, either form, including constant of integration

(ii)
NB different orderings of the columns of P and D will give the terms in different orders here.
M1: Uses their **P** and **D** to transform system into equation in u, v and w (may be implied).
M1: Forms the solution for u, v and w using their eigenvalues.
M1: Reverses the substitution (multiplies by their **P**) to get solution for $\dot{x}$, $\dot{y}$ and $\dot{z}$.
A1: Correct answer, in matrix form or as separate equations – award when first seen and isw.