GCE A level Further Mathematics (9FM0) – Shadow Paper (Set 1)

9FM0-01 Core Pure Mathematics 1

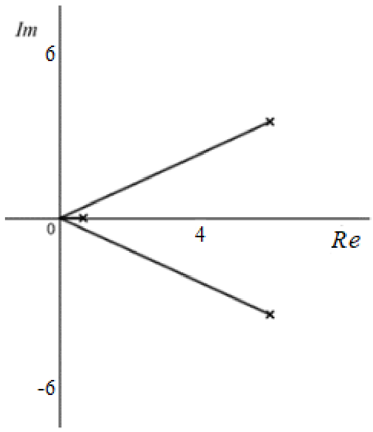
October 2020 Shadow Paper mark scheme

Please note that this mark scheme is not the one used by examiners for making scripts. It is intended more as a guide, indicating where marks are given for correct answers. As such, it may not show follow-through marks (marks that are awarded despite errors being made) or special cases.

It should also be noted that for many questions, there may be alternative methods of finding correct solutions that are not shown here – they will be covered in the formal mark scheme from the original paper.

This document is intended for guidance only and may differ significantly from the examiners' final mark scheme for the original paper which was published in December 2020.

Guidance on the use of codes within this document
M1 – method mark. This mark is generally given for an appropriate method in the context of the question. This mark is given for showing your working and may be awarded even if working is incorrect. A1 – accuracy mark. This mark is generally given for a correct answer following correct working. B1 – working mark. This mark is usually given when working and the answer cannot easily be separated. Some questions require all working to be shown; in such questions, no marks will be given for an answer with no working (even if it is a correct answer).

Question	Scheme	Marks	AOs
1(a)	$\beta = 5 + 3\sqrt{3}\,\mathrm{i}$ is also a root	B1	1.2
	$\alpha\beta = 52, \alpha + \beta = 10$	B1	1.1b
	$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{270}{5}$	M1	1.1b
	$\alpha\gamma + \beta\gamma = \frac{270}{5} - 52 = \gamma(\alpha + \beta) = 10\gamma \Rightarrow 10\gamma = 2 \Rightarrow \gamma = ...$	M1	3.1a
	$\gamma = \frac{1}{5}$	A1	2.2a
		B1	1.1b
		B1ft	1.1b
		(7)	
	(a) Alternative:		
	$\beta = 5 + 3\sqrt{3}\,\mathrm{i}$ is also a root	B1	1.2
	$\left(z - \left(5 + 3\sqrt{3}\,\mathrm{i}\right)\right)\left(z - \left(5 - 3\sqrt{3}\,\mathrm{i}\right)\right) = z^2 - 10z + 52$	B1	1.1b
	$\mathrm{f}(z) = \left(z^2 - 10z + 52\right)(5z + a) = 5z^3 + az^2 - 50z^2 - 10az + 260z + 52a$	M1	1.1b
	$\Rightarrow 260 - 10a = 270 \Rightarrow a = -1$	M1	3.1a
	$\gamma = \frac{1}{5}$	A1	2.2a
	Then B1 B1ft as above		
		(7)	
(b)	$5 - 3\sqrt{3}\,\mathrm{i} + 5 + 3\sqrt{3}\,\mathrm{i} + \frac{1}{5} = -\frac{p}{5} \Rightarrow p = ...$ or $\left(5 - 3\sqrt{3}\,\mathrm{i}\right)\left(5 + 3\sqrt{3}\,\mathrm{i}\right) \times \frac{1}{5} = -\frac{q}{5} \Rightarrow q = ...$	M1	3.1a
	$p = -51$ **or** $q = -52$	A1	1.1b
	$p = -51$ **and** $q = -52$	A1	1.1b
		(3)	

	(b) Alternative:		
	$\mathrm{f}(z)=(z^2-10z+52)(5z+a)=5z^3+pz^2+270z+q$ $\Rightarrow p=\ldots, q=\ldots$	M1	3.1a
	$p=-51$ **or** $q=-52$	A1	1.1b
	$p=-51$ **and** $q=-52$	A1	1.1b
		(3)	
			(10 marks)

Notes

(a)
B1: Identifies the correct complex conjugate as another root
B1: Correct values for the sum and product for the conjugate pair
M1: Correct application of the pair sum
M1: Identifies a complete and correct strategy for identifying the third root
A1: Deduces the correct third root
B1: $5\pm3\sqrt{3}\,\mathrm{i}$ plotted correctly
B1ft: Their real root plotted correctly
Alternative:
B1: Identifies the correct complex conjugate as another root
B1: Correct quadratic factor obtained
M1: Expands their quadratic×(5z + "a")
M1: Compares z coefficients to establish the value of "*a*"
A1: Deduces the correct third root
B1: $5\pm3\sqrt{3}\,\mathrm{i}$ plotted correctly
B1ft: Their real root plotted correctly
(b)
M1: Correct strategy used for identifying at least one of *p* or *q*
A1: At least one value correct
A1: Both values correct
Alternative:
M1: Correct strategy by expanding their quadratic and linear factors to identifying at least one of *p* or *q*
A1: At least one value correct
A1: Both values correct

Question	Scheme	Marks	AOs
2(a)	Because the upper limit is infinity	B1	2.4
		(1)	
(b)	$\dfrac{9}{x(3x-5)} = \dfrac{A}{x} + \dfrac{B}{3x-5} \Rightarrow A = ..., B = ...$	M1	3.1a
	$\dfrac{9}{x(3x-5)} = \dfrac{27}{5(3x-5)} - \dfrac{9}{5x}$	A1	1.1b
	$\int_3^\infty \dfrac{9}{3x-5}dx = \left(\int_3^\infty \left(\dfrac{27}{5(3x-5)} - \dfrac{9}{5x}\right)dx = \dfrac{9}{5}\int_3^\infty \left(\dfrac{3}{(3x-5)} - \dfrac{1}{x}\right)dx\right)$ $= \dfrac{9}{5}\left[\ln(3x-5) - \ln(x)\right]$	A1ft	1.1b
	$\dfrac{9}{5}\left[\ln(3x-5) - \ln(x)\right] = \dfrac{9}{5}\ln\dfrac{3x-5}{x}$	M1	2.1
	$\lim_{x\to\infty}\left\{\dfrac{9}{5}\ln\dfrac{3x-5}{x}\right\} = \dfrac{9}{5}\ln 3$	B1	2.2a
	$\Rightarrow \int_3^\infty \dfrac{9}{3x-5}\text{d}x = \dfrac{9}{5}\left[\ln 3 - \ln\dfrac{4}{3}\right] = \dfrac{9}{5}\ln\dfrac{9}{4}$	A1oe	1.1b
		(6)	
		(7 marks)	

Notes

(a)

B1: Correct explanation

(b)

M1: Selects the correct form for partial fractions and proceeds to find values for *A* and *B*

A1: Correct constants or partial fractions

A1ft: $\int\left(\dfrac{p}{5(3x-5)} - \dfrac{q}{5x}\right)dx = \dfrac{p/3}{5}\ln(3x-5) - \dfrac{q}{5}\ln(x)$

M1: Combines logs correctly

B1: Correct upper limit for $x \to \infty$ by recognising the dominant terms. (Simply replacing x with ∞ scores B0)

A1: Deduces the correct value for the improper integral in the correct form [note other potential correct solutions eg $\dfrac{9}{5}\ln\dfrac{9}{4} = \dfrac{9}{5}\ln\left(\dfrac{3}{2}\right)^2 = \dfrac{18}{5}\ln\dfrac{3}{2}$ but do not accept $\dfrac{9}{5}\ln\left(\dfrac{3}{2}\right)^2$ as not in correct form as asked in question]

Question	Scheme	Marks	AOs
3	$3+2\sin\theta=2\Rightarrow\sin\theta=-\frac{1}{2}\Rightarrow\theta=\ldots$	M1	3.1a
	$\theta=\frac{7\pi}{6}\left(\text{or}\ \frac{11\pi}{6}\right)$	A1	1.1b
	Use of $\frac{1}{2}\int r^2 d\theta$ (oe) for minor segment of circle or… $\frac{1}{2}\int(3+2\sin\theta)^2 d\theta$	M1	1.1a
	Correct strategy e.g. $A=\left\lvert\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\left[2^2-(3+2\sin\theta)^2\right]\mathrm{d}\theta\right\rvert$	M1	3.1a
	$A=\left\lvert\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\left[4-\left(9+12\sin\theta+4\sin^2\theta\right)\right]\mathrm{d}\theta\right\rvert$ $=\left\lvert-\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\left[5+12\sin\theta+4\sin^2\theta\right]\mathrm{d}\theta\right\rvert$	A1	1.1b
	$\int\sin^2\theta\,\mathrm{d}\theta=\frac{1}{2}\int(1-\cos 2\theta)\,\mathrm{d}\theta$	M1	3.1a
	$A=\left\lvert-\frac{1}{2}\left[7\theta-12\cos\theta-\sin 2\theta\right]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\right\rvert$	A1	1.1b
	$A=\left\lvert-\frac{1}{2}\left\{\left(\frac{77\pi}{6}-6\sqrt{3}+\frac{\sqrt{3}}{2}\right)-\left(\frac{49\pi}{6}+6\sqrt{3}-\frac{\sqrt{3}}{2}\right)\right\}\right\rvert=\ldots$ $\left(=\frac{11}{2}\sqrt{3}-\frac{7}{3}\pi\right)$	M1	2.1
	$=\frac{1}{6}\left(33\sqrt{3}-14\pi\right)$	A1	1.1b
		(9)	
			(9 marks)

Notes

M1: Realises that the angles at the intersection are required and solves $C_1 = C_2$ to obtain a value for θ

A1: Correct value for θ (other equivalent values are acceptable e.g. $\theta=-\frac{\pi}{6}\left(\text{or}\ -\frac{5\pi}{6}\right)$)

M1: Evidence of the use of a correct polar area formula on either curve or area of minor segment of the circle

M1: Uses a correct strategy to find the required area

A1: Correct expansions for both curves (may be unsimplified)

M1: Selects the correct strategy by applying the correct double angle identity in order to reach an integrable form A1: Correct integration M1: Applies limits correctly to their integration and combines terms A1: Correct area

Question	Scheme	Marks	AOs
4(a)	$x = 1 + 3\lambda - \mu$ $y = -5 + \lambda + 2\mu$ $z = 4 - 2\lambda + \mu$ $\Rightarrow$ $2x + y = -3 + 7\lambda$ $2x + 2z = 10 + 2\lambda$	M1 A1	3.1a 1.1b
	$4x + 2y - 14x - 14z = -6 - 70$ $5x - y + 7z = 38$	M1 A1	1.1b 2.5
		(4)	
(b)	$\frac{x+1}{3} = \frac{2-y}{2} = \frac{z-1}{4} \Rightarrow \mathbf{r} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ $5(-1 + 3t) - (2 - 2t) + 7(1 + 4t) = 38 \Rightarrow t = \ldots$	M1	3.1a
	$t = \frac{38}{45} \Rightarrow \mathbf{r} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \frac{38}{45}(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$	M1	1.1b
	$\left(\frac{23}{15}, \frac{14}{45}, \frac{197}{45}\right)$	A1	1.1b
		(3)	
(c)	$(5\mathbf{i} - \mathbf{j} + 7\mathbf{k})(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 15 + 1 + 14 = 30$ $30 = \sqrt{75}\sqrt{14}\cos\theta \Rightarrow \theta = \ldots$	M1	1.1b
	$\theta = 22.2^{\circ}$, $\theta =$ awrt 22°	A1	1.1b
		(2)	
			(9 marks)

Notes

(a)
M1: Uses the component form to eliminate one of the scalar parameters
A1: Two correct equations with one parameter eliminated
M1: Forms a Cartesian equation
A1: Correct Cartesian equation (accept any equivalent form)
(b)
M1: Correctly interprets the Cartesian form to give a parametric form and substitutes this into their Cartesian equation and proceeds to find a value for their parameter
M1: Substitutes their parameter value back into the parametric form of the line
A1: Correct coordinates
(c)
M1: Complete and correct scalar product method leading to a value for θ
A1: Correct angle

Question	Scheme	Marks	AOs
5(a)	$\dfrac{d^2x}{dt^2} = -7\dfrac{dx}{dt} + 10\dfrac{dy}{dt}$	B1	1.1b
	$= -7\dfrac{dx}{dt} + 10(-4x+5y-3)$ $= -7\dfrac{dx}{dt} - 40x + 50y - 30$ $= -7\dfrac{dx}{dt} - 40x + 5\left(\dfrac{dx}{dt} + 7x + 20\right) - 30$	M1	2.1
	$\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 70 *$	A1*	1.1b
		(3)	
(b)	$m^2 + 2m + 5 = 0 \Rightarrow m = ...$	M1	3.4
	$m = -1 \pm 2i$	A1	1.1b
	$m = \alpha \pm \beta i \Rightarrow x = e^{\alpha t}(A\cos \beta t + B\sin \beta t) = ...$	M1	3.4
	$x = e^{-t}(A\cos 2t + B\sin 2t)$	A1	1.1b
	PI: Try $x = k \Rightarrow 5k = 70 \Rightarrow k = 14$	M1	3.4
	$GS: x = e^{-t}(A\cos 2t + B\sin 2t) + 14$	A1ft	1.1b
		(6)	
(c)	$\dfrac{dx}{dt} = e^{-t}(2B\cos 2t - 2A\sin 2t) - e^{-t}(A\cos 2t + B\sin 2t)$	B1ft	1.1b
	$y = \dfrac{1}{10}\left(\dfrac{dx}{dt} + 7x + 20\right) = ...$	M1	3.4
	$= \dfrac{1}{10}\left\{e^{-t}\left[2(3A+B)\cos 2t + 2(3B-A)\sin 2t\right] + 118\right\}$	A1	1.1b
		(3)	
(d)	$t = 0, x = 6 \Rightarrow 6 = A + 14 \Rightarrow A = -8$	M1	3.1b
	$t = 0, y = 7.2 \Rightarrow 7.2 = 0.2(-24+B) + 11.8 \Rightarrow B = 1$	M1	3.3
	$x = e^{-t}(\sin 2t - 8\cos 2t) + 14$	A1	2.2a
	$y = \dfrac{1}{5}e^{-t}(11\sin 2t - 23\cos 2t + 59)$	A1	2.2a
		(4)	
(e)	When $t > 5$, the amount of enzyme X and the amount of enzyme Y remain (approximately) constant at 14 and 11.8 respectively, which suggests that the biological reaction has stopped. This supports the scientist's claim.	B1	3.5a
		(1)	
			(17 marks)

Notes
(a) B1: Differentiates the first equation with respect to t correctly M1: Uses the second equation to eliminate y and proceeds to printed answer A1*: Achieves the printed answer with no errors (b) M1: Uses the model to form and solve the auxiliary equation A1: Correct roots of the AE M1: Uses the model to form the complementary function A1: Correct CF M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI A1ft: Combines their CF and PI to give x in terms of t (c) B1ft: Correct differentiation of their x. Follow through their $e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$ M1: Uses the model and their answer to part (b) to give y in terms of t A1: Correct equation (d) M1: Realises the need to use the initial conditions in the equation for x M1: Realises the need to use the initial conditions in the equation for y to find both unknown constants A1: Deduces the correct equation for x A1: Deduces the correct equation for y (e) B1: Realises that, for values of $t > 5$, the amounts of compounds X and Y present do not vary, which supports the claim

Question	Scheme	Marks	AOs
6(i)	When $n = 1$, $\sum_{r=1}^{1}(2r-1)^2 = 1^2 = 1$ $\frac{1}{3}\times1\times(2\times1-1)(2\times1+1) = \frac{1}{3}\times1\times1\times3 = 1$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$	M1	2.4
	$\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1) + [2(k+1)-1]^2$ $\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$	M1	2.1
	$= \frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$ $= \frac{1}{3}(2k+1)(2k^2-k+6k+3)$ $= \frac{1}{3}(2k+1)(2k^2+5k+3)$ $= \frac{1}{3}(2k+1)(2k+3)(k+1)$	A1	1.1b
	$\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}(\underline{k+1})(2[\underline{k+1}]-1)(2[\underline{k+1}]+1)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n.	A1	2.4
		(6)	
(ii) Way 1	When $n = 1$, $3^1 + 7^1 = 10$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^k + 7^k$ is divisible by 10	M1	2.4
	$f(k+2) = 3^{k+2} + 7^{k+2}$	M1	2.1
	$= 9\times3^k + 49\times7^k = 9\times3^k + 9\times7^k + 40\times7^k$ $= 9f(k) + 10\times(4\times7^k)$	A1 A1	1.1b 1.1b
	<u>If true for $n = k$</u> then <u>true for $n = k + 2$</u>, <u>true for $n = 1$</u> so <u>true for all (positive odd integers) n</u>	A1	2.4
		(6)	
(ii) Way 2	When $n = 1$, $3^1 + 7^1 = 10$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^k + 7^k$ is divisible by 10	M1	2.4

	$f\left(k+2\right)-f\left(k\right)=3^{k+2}+7^{k+2}-3^{k}-7^{k}$	M1	2.1
	$=8\times3^{k}+48\times7^{k}$	A1	1.1b
	$f\left(k+2\right)=8f\left(k\right)+10\times\left(4\times7^{k}\right)$	A1	1.1b
	<u>If true for $n=k$</u> then <u>true for $n=k+2$</u>, <u>true for $n=1$</u> so <u>true for all (positive odd integers) n</u>	A1	2.4
		(6)	
(12 marks)			

Notes

(i)
B1: Shows the statement is true for $n = 1$
M1: Makes an assumption statement that assumes the result is true for $n = k$
M1: Attempts to add the $(k + 1)^{th}$ term to the assumed result
A1: Correct expression with at least one correct linear factor
A1: Obtains a fully correct expression in terms of $k + 1$
A1: Correct complete conclusion with all ideas conveyed at the end or as a narrative
(ii) **Way 1**
B1: Shows that f(1) = 10
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 2)
A1: Correctly obtains 9f(k) **or** 40×7^k
A1: Reaches a correct expression for f(k + 2) in terms of f(k)
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.
Way 2
B1: Shows that f(1) = 10
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 2) – f(k) or equivalent work
A1: Achieves a correct expression for f(k + 2) – f(k) in terms of f(k)
A1: Reaches a correct expression for f(k + 2) in terms of f(k)
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Question	Scheme	Marks	AOs
7(a)	$(2+t)\frac{\mathrm{dN}}{\mathrm{d}t}+N=\frac{1}{\sqrt{t}}(2+t)\Rightarrow\frac{\mathrm{dN}}{\mathrm{d}t}+\frac{N}{2+t}=t^{-\frac{1}{2}}$	B1	1.1b
	$I=e^{\int\frac{1}{2+t}\mathrm{d}t}=2+t\Rightarrow N(2+t)=\int t^{-\frac{1}{2}}(2+t)\mathrm{d}t$ $I=\mathrm{e}^{\int\frac{1}{2+t}\mathrm{d}t}=2+t\Rightarrow N(2+t)=\int t^{-\frac{1}{2}}(2+t)\mathrm{d}t$	M1	3.1b
	$N(2+t)=4t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}+c$	A1	1.1b
	$t=0,\ N=7\Rightarrow c=14$	M1	3.4
	$N=\frac{4t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}+14}{(2+t)}=\frac{4\times6^{\frac{1}{2}}+\frac{2}{3}6^{\frac{3}{2}}+14}{8}=...$	M1	1.1b
	= awrt 420 insects	A1	2.2b
		(6)	
(b)	$N=\frac{4t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}+14}{(2+t)}\Rightarrow\frac{\mathrm{dN}}{\mathrm{d}t}=\frac{(2+t)\left(2t^{-\frac{1}{2}}+t^{\frac{1}{2}}\right)-\left(4t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}+14\right)}{(2+t)^2}$	M1 A1	3.4 1.1b
	$\left(\frac{dN}{dt}\right)_{t=36}=\frac{(38)\left(2\cdot36^{-\frac{1}{2}}+36^{\frac{1}{2}}\right)-\left(4\cdot36^{\frac{1}{2}}+\frac{2}{3}36^{\frac{3}{2}}+14\right)}{(38)^2}=0.04$	M1	3.1a
	$0.04\times100=4$ insects per hour	A1	3.2a
		(4)	
	(b) Alternative:		
	$P=\frac{4t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}+14}{(2+t)}=\frac{4\cdot36^{\frac{1}{2}}+\frac{2}{3}\cdot36^{\frac{3}{2}}+14}{(2+36)}$	M1	3.4
	$=\frac{91}{19}$	A1	1.1b
	$(2+t)\frac{\mathrm{d}N}{\mathrm{d}t}+N=\frac{1}{\sqrt{t}}(2+t)\Rightarrow38\frac{\mathrm{dN}}{\mathrm{d}t}+\frac{91}{19}=\frac{1}{6}\times38\Rightarrow\frac{\mathrm{dN}}{\mathrm{d}t}=\frac{44}{1083}$	M1	3.1a
	$\frac{44}{1083}\times100=\frac{4400}{1083}(=\text{awrt }4)$ insects per hour	A1	3.2a
		(4)	
(c)	E.g. The number of insects increases indefinitely, albeit quite slowly, which is not realistic	B1	3.5b
		(1)	
			(11 marks)

Notes

(a)
B1: A correct rearrangement (may be implied by subsequent work)

M1: Uses the model to find the integrating factor and attempts the solution of the differential equation
A1: Correct solution
M1: Interprets the initial conditions to find the constant of integration
M1: Uses their solution to the problem to find the population after 6 hours
A1: Correct number of insects
(b)
M1: Realises the need to differentiate the model and uses an appropriate method to find the derivative
A1: Correct differentiation
M1: Uses $t = 36$ in their dN/dt
A1: Correct answer with correct units
Alternative:
M1: Substitutes $t = 36$ into their N
A1: Correct value for N
M1: Uses $t = 36$ and their N to find a value for dN/dt
A1: Correct answer with correct units
(c)
B1: Suggests a suitable limitation