GCE A Level Further Mathematics (9FM0) – Shadow Paper (Set 1)
9FM0-01 Core Pure Mathematics 1

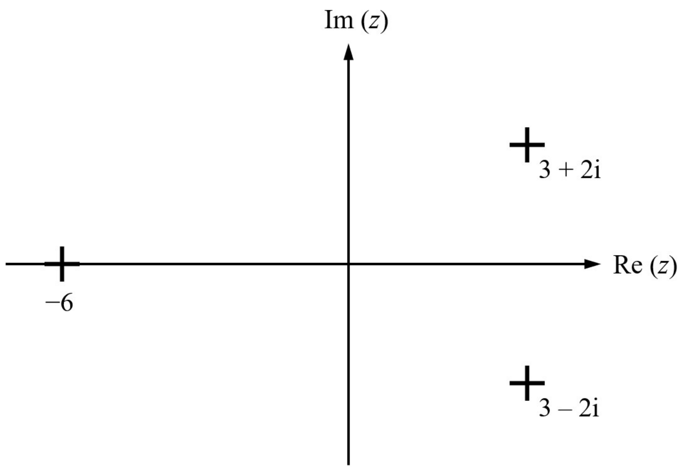
June 2022 Shadow Paper mark scheme

Please note that this mark scheme is not the one used by examiners for making scripts. It is intended more as a guide, indicating where marks are given for correct answers. As such, it may not show follow-through marks (marks that are awarded despite errors being made) or special cases.

It should also be noted that for many questions, there may be alternative methods of finding correct solutions that are not shown here – they will be covered in the formal mark scheme from the original paper.

This document is intended for guidance only and may differ significantly from the examiners' final mark scheme for the original paper, which was published in August 2022.

Guidance on the use of codes within this document
M1 – method mark. This mark is generally given for an appropriate method in the context of the question. This mark is given for showing your working and may be awarded even if working is incorrect. A1 – accuracy mark. This mark is generally given for a correct answer following correct working. B1 – working mark. This mark is usually given when working and the answer cannot easily be separated. Some questions require all working to be shown; in such questions, no marks will be given for an answer with no working (even if it is a correct answer).

Question	Scheme	Marks	AOs
1(a)	$3+2\text{i}$	B1	1.1b
		(1)	
(b) (i)	$z^* = 3-2\text{i}$ so $z+z^*=6$, $zz^*=13$ $z+z^*+\alpha=0 \Rightarrow \alpha = ...$ or $\alpha zz^* = -78 \Rightarrow \alpha = -\frac{78}{"13"} = ...$ or $z^2-(\text{sum roots})z+(\text{product roots})=0$ or $(z-(3+2\text{i}))(z-(3-2\text{i}))=...$ $\Rightarrow (z^2-6z+13)\underline{(z+6)} \Rightarrow z = ...$	M1	3.1a
(ii)	$z = 3\pm 2\text{i}, -6$	A1	1.1b
	$(z^2-6z+13)(z+6)$ expands the brackets to find value for a Or a = pair sum = $-6(3+2\text{i}+3-2\text{i})+13=...$ Or $f(-6)\,/\,f(3\pm 2\text{i})=0 \Rightarrow ... \Rightarrow a = ...$	M1	1.1b
	$a = 23$	A1	2.2a
		(4)	
(c)		B1ft	1.1b
		(1)	
			(6 marks)

Question	Scheme	Marks	AOs
2	Solves the quadratic equation for $\sinh^2 x$ e.g. $$\left(5\sinh^2 x-4\right)\left(5\sinh^2+1\right)=0 \Rightarrow \sinh^2 x=...$$	M1	3.1a
	$$\sinh^2 x=\frac{4}{5}\left\{-\frac{1}{5}\right\}$$	A1	1.1b
	$$\sinh x=\frac{2}{5}\sqrt{5} \Rightarrow x=\ln\left[\frac{2}{5}\sqrt{5}+\sqrt{\left(\frac{2}{5}\sqrt{5}\right)^2-1}\right]$$ **Alternatively** $\sinh x=\frac{2}{5}\sqrt{5}=\frac{1}{2}\left(e^x+e^{-x}\right) \Rightarrow 5e^{2x}-4\sqrt{5}e^x+5=0$. $\Rightarrow e^x=\sqrt{5}$ or $\frac{-1}{\sqrt{5}} \Rightarrow x=...$	M1	1.1b
	$$x=\pm\frac{1}{2}\ln 5$$	A1	2.2a
			(4 marks)

Question	Scheme	Marks	AOs
3(a)	$\frac{dy}{dx} + y\cot x = e^x \operatorname{cosec} x$ $\text{IF} = e^{\int \cot x\, dx} = e^{\ln \sin x} = \sin x \Rightarrow \sin x \frac{dy}{dx} + y \sin x \cot x = e^x$ $\Rightarrow y \sin x = \int e^x \, dx$	M1	3.1a
	$y \sin x = e^x \left(+c\right)$	A1	1.1b
	$y = \frac{e^x + c}{\sin x}$	A1	1.1b
		(3)	
(b)	$x = \frac{\pi}{2},\ y = 0 \Rightarrow c = \ldots \left\{ -e^{\frac{\pi}{2}} or -4.81\ldots \right\}$	M1	3.1a
	$y = \frac{e^x - e^{\frac{\pi}{2}}}{\sin x}$	M1	1.1b
	Require $e^x - e^{\frac{\pi}{2}} = 0$, hence only solution to $y = 0$ is $x = \frac{\pi}{2}$	A1	1.1b
		(3)	
(6 marks)			

Question	Scheme	Marks	AOs
4(a)	Applies $\ln\left(\frac{2r+1}{2r-1}\right)=\ln(2r+1)-\ln(2r-1)$ to the problem in order to apply differences.	M1	3.1a
	$\sum_{r=1}^{n}\left(\ln(2r+1)-\ln(2r-1)\right)$ $=\left(\ln(3)-\ln(1)\right)+\left(\ln(5)-\ln(3)\right)+\left(\ln(7)-\ln(5)\right)+\ldots$ $+\left(\ln(2n)-\ln(2n-2)\right)+\left(\ln(2n+1)-\ln(2n-1)\right)$	dM1	1.1b
	$\ln(2n+1)-\ln 1$	A1	1.1b
	$\ln(2n+1)$ * cso	A1 *	2.1
		(4)	
(b)	$\sum_{r=51}^{100}\ln\left(\frac{r+1}{r-1}\right)=\sum_{r=1}^{100}\ln\left(\frac{2r+1}{2r-1}\right)-\sum_{r=1}^{50}\ln\left(\frac{2r+1}{2r-1}\right)=\ln(201)-\ln(101)$	M1	1.1b
	$\sum_{r=51}^{100}\ln\left(\frac{2r+1}{2r-1}\right)^{45}=45\ln\left(\frac{201}{101}\right)$	M1	3.1a
	$=45\ln\left(\frac{201}{101}\right)$	A1	1.1b
		(3)	
			(7 marks)

Question	Scheme	Marks	AOs
5(a)	$\det(\boldsymbol{M}) = 1(-4-3a) - 2(0-9) - a(0+3)$	M1	1.1b
	$\det(\boldsymbol{M}) = 14 \neq 0$ therefore, non-singular for all values of a	A1	2.4
		(2)	
(b)	Transposes the matrix (M^T) $$\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & a \\ a & 3 & 4 \end{pmatrix}$$	M1	1.1b
	Finds the matrix of cofactors (adjM). $$\begin{pmatrix} -3a-4 & a^2-8 & a+6 \\ 9 & 4-3a & -3 \\ 3 & 6-a & -1 \end{pmatrix}$$	M1	1.1b
	$$= \frac{1}{14}\begin{pmatrix} -3a-4 & a^2-8 & a+6 \\ 9 & 4-3a & -3 \\ 3 & 6-a & -1 \end{pmatrix}$$	M1 A1	1.1b 2.1
		(4)	
			(6 marks)

Question	Scheme	Marks	AOs
6(a)	$\frac{3x^2+5x+5}{(x+2)(x^2+3)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+3}\Rightarrow 3x^2+5x+5=A(x^2+3)+(Bx+C)(x+2)$	M1	1.1b
	e.g. $x=-2\Rightarrow A=...,\ x=0\Rightarrow C=...,$ coeff $x^2\Rightarrow B=...$ or Compares coefficients and solves to find values for A, B and C $2=A+B, 3=B+C, 6=4A+C$.	dM1	1.1b
	$A=1,\ B=2,\ C=1$	A1	1.1b
		(3)	
(b)	$\int_0^1 \frac{1}{x+2}+\frac{2x+1}{x^2+3}\,dx=\int_0^1 \frac{1}{x+2}+\frac{2x}{x^2+3}+\frac{1}{x^2+3}\,dx$ $=\left[\alpha\ln(x+2)+\beta\ln(x^2+3)+\lambda\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$	M1	3.1a
	$=\left[\ln(x+2)+\ln(x^2+3)+\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$		2.1
	$=\left[\ln(3)+\ln(4)+\frac{1}{\sqrt{3}}\frac{\pi}{6}\right]-\left[\ln(2)+\ln(3)+\arctan(0)\right]=$ $=\left[\ln(4)-\ln(2)+\frac{1}{\sqrt{3}}\frac{\pi}{6}\right]=2\ln 2-\ln 2+\frac{1}{\sqrt{3}}\frac{\pi}{6}$	dM1	2.1
	$\ln(2)+\frac{\sqrt{3}\pi}{18}$	A1	2.2a
		(4)	
(7 marks)			

Question	Scheme	Marks	AOs
7(a)	$z^* = a - bi$ then $zz^* = (a+bi)(a-bi) = ...$	M1	1.1b
	$zz^* = a^2 + b^2$ therefore, a real number	A1	2.4
		(2)	
(b)	$\frac{z}{z^*} = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{(a^2-b^2)+2abi}{a^2+b^2} = \frac{1}{7} + \frac{4\sqrt{3}i}{7}$ or . $\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{7} \Rightarrow z^2 = 1 + 4\sqrt{3}i$ or $a+bi = \left(\frac{1}{7} + \frac{4\sqrt{3}i}{7}\right)(a-bi) = ..+..i$.	M1	1.1b
	Forms two equations from $a^2+b^2 = 7$ or $\frac{a^2-b^2}{a^2+b^2} = \frac{1}{7}$ or $\frac{a^2-b^2}{7} = \frac{1}{7}$ or $\frac{2ab}{7} = \frac{4\sqrt{3}}{7}$ or $\frac{2ab}{a^2+b^2} = \frac{4\sqrt{3}}{7}$ or $a = \frac{1}{7}a + \frac{4\sqrt{3}}{7}b$ oe	M1 A1	3.1a 1.1b
	Solves the equations simultaneously g. $a^2+b^2 = 7$ and $a^2-b^2 = 1$ leading to a value for a or b	dM1	1.1b
	$z = \pm\left(2 + \sqrt{3}i\right)$		2.2a
		(5)	
			(7 marks)
(b) **Alt**	$\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{7} \Rightarrow z^2 = 1 + 4\sqrt{3}i$ or let arg $z = \theta$. then $\frac{z}{z^*} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta} = \cos 2\theta + \text{i}\sin 2\theta$	M1	1.1b
	$z^2 = 7(\cos\alpha + \text{i}\sin\alpha)$ where tan $\alpha = 4\sqrt{3} \Rightarrow z = \pm\sqrt{7}\left(\cos\frac{1}{2}\alpha + \text{i}\sin\frac{1}{2}\alpha\right)$ Or $\cos 2\theta + \text{i}\sin 2\theta = \frac{1}{7} + \frac{4\sqrt{3}i}{7} \Rightarrow 2cos^2\theta - 1 = \frac{1}{7}, 2\sin\theta\cos\theta = \frac{4\sqrt{3}}{7}$	M1 A1	1.1b 1.1b
	$\cos\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1+\cos\alpha)} = \sqrt{\frac{1}{2}\left(1+\frac{1}{7}\right)} = ...$ and $\sin\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1-\cos\alpha)} = \sqrt{\frac{1}{2}\left(1-\frac{1}{7}\right)} = ...$ or $\Rightarrow \cos\theta = \frac{2}{\sqrt{7}}, \sin\theta = \frac{\sqrt{3}}{\sqrt{7}}, r = \|z\| = \sqrt{zz^*} = \sqrt{7}$	dM1	3.1a
	$z = \pm\left(2 + \sqrt{3}\text{i}\right)$.	A1	2.2a
		(5)	

Question	Scheme	Marks	AOs
8(a)	$\left(z+\frac{1}{z}\right)^4=16\cos^4\theta$	B1	2.1
	$\left(z+\frac{1}{z}\right)^4=z^4+4\left(z^3\right)\left(\frac{1}{z}\right)+6\left(z^2\right)\left(\frac{1}{z^2}\right)+4\left(z\right)\left(\frac{1}{z^3}\right)+\left(\frac{1}{z^4}\right)$	M1	2.1
	$=\left[z^4+\frac{1}{z^4}\right]+4\left[z^2+\frac{1}{z^2}\right]+6$.	A1	1.1b
	Uses $z^n+\frac{1}{z^n}=2\cos n\theta$ $\{16\cos^4\theta\}=2\cos 4\theta+8\cos 2\theta+6$	M1	2.1
	$8\cos^4\theta=\cos 4\theta+4\cos 4\theta+3$ * cso	A1 *	1.1b
		(5)	
(b)	$\boldsymbol{H}=2$	B1	3.3
		(1)	
(c)	$\text{vol}=\left\{\frac{1}{2}\right\}\pi\int\left(2\cos^2\left(\frac{x}{4}\right)\right)^2\,\mathrm{d}x$	ft	3.4
	$\text{vol}=\{2\pi\}\int\cos^4\left(\frac{x}{4}\right)\mathrm{d}x$ $=\{2\pi\}\int\frac{1}{16}\left(2\cos\left(\frac{4x}{4}\right)+8\cos\left(\frac{2x}{4}\right)+6\right)\mathrm{d}x=\ldots$	M1	1.1b
	$=\{2\pi\}\left[\frac{1}{16}\underline{\left(2\sin(x)+8\sin\left(\frac{x}{2}\right)+6x\right)}\right]$	A1	1.1b
	$=2\times 2\pi\left[\frac{1}{16}\left(2\sin(4)+8\sin\left(\frac{4}{2}\right)+(6\times 4)\right)\right]=\ldots$ or $=2\pi\left[\begin{array}{c}\frac{1}{16}\left(2\sin(4)+8\sin\left(\frac{4}{2}\right)+(6\times 4)\right)\\-\frac{1}{16}\left(2\sin(-4)+8\sin\left(\frac{-4}{2}\right)+(6\times(-4))\right)\end{array}\right]=\ldots$	dM1	3.4
	$=29.09$	A1	1.1b
		(5)	

(d)	The equation of the curve may not be suitable The measurements may not be accurate The paperweight may not be smooth	B1	3.5b
		(1)	
			(12 marks)

Question	Scheme	Marks	AOs
9(i) (a)	E. g. ● Because the interval being integrated over is unbounded. ● sinh x is undefined at the limit of ∞ ● the upper limit is infinite	B1	1.2
		(1)	
(i) (b)	$\int_0^{\infty} \sinh x \, dx = \lim_{t\to\infty} \int_0^t \sinh x \, dx$ or $\lim_{t\to\infty} \int_o^t \frac{1}{2}\left(e^x - e^{-x}\right) dx$	B1	2.5
	$\int_0^t \sinh x \, dx = [\cosh x]_0^t = \cosh t - 1$ or $\frac{1}{2}\int_0^t e^x - e^{-x} \, dx = \frac{1}{2}\left[e^x + e^{-x}\right]_0^t = \frac{1}{2}\left[e^t + e^{-t}\right] \left(-\frac{1}{2}\left[e^0 + e^0\right]\right)$	M1	1.1b
	When $t \to \infty$ $e^t \to \infty$ and $e^{-t} \to 0$. therefore the integral is divergent. (In the limit, $t \to \infty, e^t - 1 \to \infty$.)	A1	2.4
		(3)	
(ii)	$3\sinh x = q\cosh x \Rightarrow \tanh x = \frac{q}{3}$ or $3\tanh x = q$ **Alternative** $\frac{3}{2}\left(e^x - e^{-x}\right) = \frac{q}{2}\left(e^x + e^{-x}\right) \Rightarrow 3e^x - 3e^{-x} = qe^x + qe^{-x}$ $e^{2x}(3-q) = q + 3 \Rightarrow e^{2x} = \frac{q+3}{3-q}$	M1	3.1a
	$\left\{-1 < \frac{q}{3} < 1 \Rightarrow\right\} -3 < q < 3$		2.2a
		(2)	
			(6 marks)

Question	Scheme		Marks	AOs
10(a)(i)	$\frac{d\theta}{dt} = \alpha \sin 2t + \beta t \cos 2t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 2t + \gamma t \sin 2t$	Let $\theta = \lambda t \sin 2t$ $\frac{d\theta}{dt} = \alpha \sin 2t + \beta t \cos 2t$. and $\frac{d^2\theta}{dt^2} = \delta \cos 2t + \gamma t \sin 2t$	M1	1.1b
	$\frac{d\theta}{dt} = \frac{1}{4}\sin 2t + \frac{1}{2}t \cos 2t$. and $\frac{d^2\theta}{dt^2} = \cos 2t - t \sin 2t$.	$\frac{d\theta}{dt} = \lambda \sin 2t + 2\lambda t \cos 2t$ and $\frac{d^2\theta}{dt^2} = 2\lambda \cos 2t + 2\lambda \cos 2t - 4\lambda t \sin 2t$ $= 4\lambda \cos 2t - 4\lambda t \sin 2t$	A1	1.1b
	$\cos 2t - t\sin 2t + t \sin 2t = ...$		dM1	3.4
	$= \cos 2t$ so PI is $\theta = \frac{1}{4}t \sin 2t$ *	$\theta = \frac{1}{4}t \sin 2t$ *	A1*	2.1
			(4)	
(a)(ii)	$m^2 + 4 = 0 \Rightarrow m = \pm 2i$		M1	1.1b
	$\theta = A\cos 2t + B \sin 2t$		A1	1.1b
	$(\theta =) CF + PI$		dM1	1.1b
	$\theta = A\cos 2t + B\sin 2t + \frac{1}{4}t \sin 2t$		A1	1.1b
			(4)	
(b)	$t = 0, \theta = \frac{\pi}{2} \Rightarrow A = ...\left\{\frac{\pi}{2}\right\}$		M1	3.4
	$t = 0, \frac{d\theta}{dt} = -2A\sin 2t + 2B\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{2}t\cos 2t = 0$ $t = 0 \Rightarrow \frac{d\theta}{dt} = 0 + 2B + 0 + 0$ $\Rightarrow B = ...\{0\}$		M1	3.4
	$\alpha = \frac{\pi}{2}\cos(2\times 6) + \frac{1}{4}(6)\sin(2\times 6) = ...$		ddM1	1.1b
	$\alpha = \pm$awrt 0.521		A1	3.4
			(4)	

(c)	0.521 is close to 0.49 so a good model (at $t = 6$)	B1ft	3.5a
		(1)	
(d)	$\frac{d^2\theta}{dt^2} + 4\theta = 0$ oe	B1	3.5c
		(1)	
			(14 marks)