

1YGB - FP3 PAGE N - QUESTION 1

FORMING A TABLE OF VALUES FOR THE INTEGRAND, USING 7 ORDINATES (6 STRIPS)

$x$	1	1.25	1.50	1.75	2	2.25	2.5
$\sqrt{x^3+1}$	1.4142	1.7185	2.0917	2.5218	3	3.5200	4.0774

FIRST    ODD    EVEN    ODD    EVEN    ODD    LAST

USING SIMPSON'S RULE

$$\int_1^{2.5} \sqrt{x^3+1} dx \approx \frac{\text{"THICKNESS"}}{3} [\text{FIRST} + \text{LAST} + 2 \times \text{EVEN} + 4 \times \text{ODDS}]$$

$$\approx \frac{0.25}{3} [1.4142 + 4.0774 + 2(2.0917 + 3) + 4(1.7185 + 2.5218 + 3.5200)]$$

$$\approx \frac{1}{12} \times 46.7162...$$

$$\approx 3.8930...$$

$$\approx \underline{3.89}$$

/ /  
3 s.f.

1YGB - P3 PAPER N - QUESTION 2

DIFFERENTIATE THE O.D.E IN SUCCESSION AND EVALUATE THE DERIVATIVES AT  $x=0$

<u>DIFFERENTIATIONS</u>	<u>EVALUATIONS</u>
	$y_0 = 2$ (GIVEN)
$y' = x^2 - y^2$	$y'_0 = x_0^2 - y_0^2$ $y'_0 = 0^2 - 2^2$ $y'_0 = -4$
$y'' = 2x - 2yy'$	$y''_0 = 2x_0 - 2y_0y'_0$ $y''_0 = 2 \times 0 - 2 \times 2 \times (-4)$ $y''_0 = 16$
$y''' = 2 - 2y'y' - 2yy''$	$y'''_0 = 2 - 2y'_0y'_0 - 2y_0y''_0$ $y'''_0 = 2 - 2(-4)^2 - 2 \times 2 \times 16$ $y'''_0 = -94$

EXPANDING AS A POWER SERIES

$$y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + o(x^4)$$

$$y = 2 + x(-4) + \frac{x^2}{2}(16) + \frac{x^3}{6}(-94) + o(x^4)$$

$$y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + o(x^4)$$

1YGB - FP3 PAPER N - QUESTION 3

As THE DENOMINATOR IS UNDER THE MODULUS SIGN, I.E IT IS NON-NEGATIVE, WE MAY MULTIPLY ACROSS

$$\Rightarrow \frac{5x-1}{|2x-3|} \geq 1$$

$$\Rightarrow 5x-1 \geq |2x-3|$$

SOLVING THE CORRESPONDING EQUATION TO OBTAIN THE CRITICAL VALUES OF THE INEQUALITY

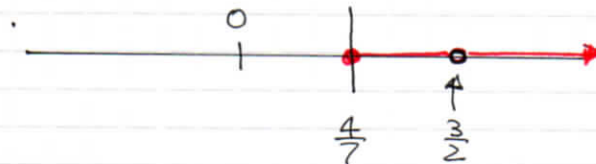
$$5x-1 = |2x-3|$$

$$\left( \begin{array}{l} 5x-1 = 2x-3 \\ 5x-1 = 3-2x \end{array} \right) \Rightarrow \left( \begin{array}{l} 3x = -2 \\ 7x = 4 \end{array} \right)$$

$$\Rightarrow x = \left\langle \begin{array}{l} \cancel{\frac{2}{3}} \\ \frac{4}{7} \end{array} \right. \text{ DOES NOT SATISFY THE ORIGINAL}$$

ONLY CRITICAL VALUE IS  $x = \frac{4}{7}$  - CHECK IF SAY 0 WORKS

$$\begin{aligned} 5(0) - 1 &= -1 \\ |2(0) - 3| &= 3 \end{aligned}$$



$$\therefore \underline{x \geq \frac{4}{7}, x \neq \frac{3}{2}}$$

- 1 -

# 1YGB - FP3 PAPER N - QUESTION 4

a) DIFFERENTIATING IMPLICITLY & FIND GRADIENT AT P

$$y^2 = 12x$$

$$2y \frac{dy}{dx} = 12$$

$$\frac{dy}{dx} = \frac{6}{y}$$

$$\left. \frac{dy}{dx} \right|_{y=6t} = \frac{6}{6t} = \frac{1}{t}$$

EQUATION OF THE TANGENT AT THE GENERAL POINT P(3t<sup>2</sup>, 6t)

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - 6t = \frac{1}{t}(x - 3t^2)$$

$$\Rightarrow yt - 6t^2 = x - 3t^2$$

$$\Rightarrow \underline{yt = x + 3t^2}$$

AS REQUIRED

b) II) START BY OBTAINING THE COORDINATES OF Q & S

• WITHIN  $x=0$

$$yt = 0 + 3t^2$$

$$y = 3t$$

$$\therefore \underline{Q(0, 3t)}$$

•  $y^2 = 12x$

$$y^2 = 4 \times 3 \times x \quad (y^2 = 4ax)$$

$$\therefore \underline{\text{FOCUS } S(3, 0)}$$

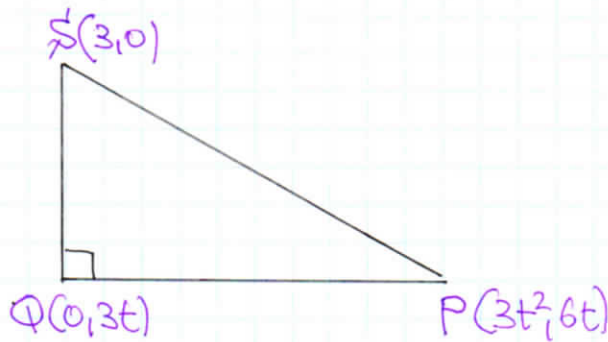
• GRADIENT PQ =  $\frac{6t - 3t}{3t^2 - 0} = \frac{3t}{3t^2} = \frac{1}{t}$  (INDEED AS EXPECTED!)

• GRADIENT SQ =  $\frac{3t - 0}{0 - 3} = \frac{3t}{-3} = -t$

AS THESE GRADIENTS ARE NEGATIVE RECIPROCAL OF ONE ANOTHER PQ IS PERPENDICULAR TO SQ

1 YGB - FP3 PAPER N - QUESTION 4

II) DRAWING A DIAGRAM TO RECORD THE INFO



- $|PQ| = \sqrt{(0-3t^2)^2 + (3t-6t)^2} = \sqrt{9t^4 + 9t^2}$
- $|SQ| = \sqrt{(3-0)^2 + (0-3t)^2} = \sqrt{9 + 9t^2}$
- $AREA = \frac{1}{2} |PQ| |SQ| = \frac{1}{2} \sqrt{9t^4 + 9t^2} \sqrt{9 + 9t^2}$   
 $= \frac{1}{2} |3t| \sqrt{t^2 + 1} \times 3\sqrt{1 + t^2}$   
 $= \underline{\underline{\frac{9}{2} |t| (t^2 + 1)}}$  // AS REQUESTED

$\sqrt{x^2} = |x|$

ALTERNATIVE FOR bii

AREA OF TRIANGLE WITH VERTICES AT  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  IS GIVEN BY

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$AREA = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 3t^2 \\ 0 & 3t & 6t \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 3 \times 3t & 1 & 1 \\ 1 & 0 & t^2 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \frac{9}{2} \left| \begin{vmatrix} t & t^2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right| \leftarrow \text{EXPANDING BY THE FIRST COLUMN}$$

$$= \frac{9}{2} |t(-t^2 - 1)| = \frac{9}{2} |-t(t^2 + 1)| = \underline{\underline{\frac{9}{2} |t| (t^2 + 1)}}$$

# 1VGB - FP3 PAPER N - QUESTION 5

## USING THE SUBSTITUTION GIVEN

$$\Rightarrow y = x V(x)$$

$$\Rightarrow \frac{dy}{dx} = 1 \times V(x) + x \frac{dV}{dx}$$

$$\Rightarrow \frac{dy}{dx} = V + x \frac{dV}{dx}$$

## SUBSTITUTING INTO THE O. D. E.

$$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$$

$$\Rightarrow V + x \frac{dV}{dx} = \frac{3x+2xV}{3xV-2x}$$

$$\Rightarrow V + x \frac{dV}{dx} = \frac{3+2V}{3V-2}$$

$$\Rightarrow x \frac{dV}{dx} = \frac{3+2V}{3V-2} - V$$

$$\Rightarrow x \frac{dV}{dx} = \frac{3+2V-3V^2+2V}{3V-2}$$

$$\Rightarrow x \frac{dV}{dx} = \frac{3+4V-3V^2}{3V-2}$$

## SEPARATING VARIABLES

$$\Rightarrow \int \frac{3V-2}{3+4V-3V^2} dV = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{3V-2}{3V^2-4V-3} dV = \int -\frac{1}{x} dx$$

$$\Rightarrow \int \frac{6V-4}{3V^2-4V-3} dV = \int -\frac{2}{x} dx$$

$$\Rightarrow \ln|3V^2-4V-3| = -2\ln|x| + \ln A$$

$$\Rightarrow \ln|3V^2-4V-3| = \ln\left(\frac{1}{x^2}\right) + \ln A$$

$$\Rightarrow \ln|3V^2-4V-3| = \ln\left|\frac{A}{x^2}\right|$$

$$\Rightarrow 3V^2-4V-3 = \frac{A}{x^2}$$

## REVERSING THE TRANSFORMATION

$$\Rightarrow 3\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) - 3 = \frac{A}{x^2}$$

$$\Rightarrow \frac{3y^2}{x^2} - \frac{4y}{x} - 3 = \frac{A}{x^2}$$

$$\Rightarrow 3y^2 - 4xy - 3x^2 = A$$

## APPLY CONDITION (1,3)

$$3 \times 3^2 - 4 \times 1 \times 3 - 3 \times 1^2 = A$$

$$27 - 12 - 3 = A$$

$$A = 12$$

$$\therefore \underline{3y^2 - 4xy - 3x^2 = 12}$$

1YGB - FP3 PAPER N - QUESTION 5

ALTERNATIVE SUBSTITUTION

$$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$$

$$\Rightarrow 3 \frac{dy}{dx} = \frac{9x+6y}{3y-2x}$$

$$\Rightarrow \frac{dv}{dx} + 2 = \frac{9x + (2v + 4x)}{v}$$

$$\Rightarrow \frac{dv}{dx} + 2 = \frac{2v + 13x}{v}$$

$$\Rightarrow \frac{dv}{dx} + 2 = \cancel{2} + \frac{13x}{v}$$

$$\Rightarrow \int v \, dv = \int 13x \, dx$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{13}{2}x^2 + C$$

$$\Rightarrow v^2 = 13x^2 + C$$

$$\Rightarrow (3y-2x)^2 = 13x^2 + C$$

Apply (1,3)  $\Rightarrow (9-2)^2 = 13 \times 1^2 + C$

$$\Rightarrow 49 = 13 + C$$

$$\Rightarrow C = 36$$

$$\Rightarrow (3y-2x)^2 = 13x^2 + 36$$

$$\Rightarrow 9y^2 - 12xy + 4x^2 = 13x^2 + 36$$

$$\Rightarrow 9y^2 - 12xy - 9x^2 = 36$$

$$\Rightarrow \underline{3y^2 - 4xy - 3x^2 = 12}$$

As before

Let $v(x) = 3y - 2x$
$\frac{dv}{dx} = 3 \frac{dy}{dx} - 2$
$3 \frac{dy}{dx} = \frac{dv}{dx} + 2$
ALSO WE HAVE
$3y = v + 2x$
$6y = 2v + 4x$

1YGB-FP3 PAPER N - QUESTION 5

ALTERNATIVE BY MULTIVARIABLE CALCULUS

$$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$$

$$\Rightarrow (3y-2x) dy = (3x+2y) dx$$

$$\Rightarrow (3x+2y) dx + (2x-3y) dy = 0$$

$$\frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy = dG$$

NOTICING THIS IS EXACT AS  $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} = 2$

HENCE WE HAVE BY DIRECT INTEGRATION

- $dG = 0 \Rightarrow G(x,y) = \text{constant}$
- $\frac{\partial G}{\partial x} = 3x+2y \Rightarrow G(x,y) = \frac{3}{2}x^2 + 2xy + f(y)$
- $\frac{\partial G}{\partial y} = 2x-3y \Rightarrow G(x,y) = 2xy - \frac{3}{2}y^2 + g(x)$

$$\therefore f(y) = -\frac{3}{2}y^2 \quad \& \quad g(x) = \frac{3}{2}x^2$$

THUS WE OBTAIN

$$G(x,y) = \text{constant}$$

$$\frac{3}{2}x^2 + 2xy - \frac{3}{2}y^2 = \text{constant}$$

$$3x^2 + 4xy - 3y^2 = \text{constant}$$

(1,3) NOW YIELDS

$$3 + 12 - 27 = \text{constant}$$

$$\text{constant} = -12$$

$$3x^2 + 4xy - 3y^2 = -12$$

$$\underline{3y^2 - 4xy - 3x^2 = 12}$$

AS BEFORE



- 1 -

1YGB - FP3 PAPER N - QUESTION 6

a) CALCULATE THE RELEVANT VECTORS FOR A CROSS PRODUCT

$$\vec{AB} = \underline{b} - \underline{a} = (9, 1, 0) - (5, 1, 3) = (4, 0, -3)$$

$$\vec{AD} = \underline{d} - \underline{a} = (-3, 8, 6) - (5, 1, 3) = (-8, 7, 3)$$

$$\text{AREA} = |\vec{AB} \wedge \vec{AD}| = \left| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 0 & -3 \\ -8 & 7 & 3 \end{vmatrix} \right| = |21, 12, 28|$$

$$= \sqrt{21^2 + 12^2 + 28^2} = \sqrt{1369} = 37$$

b) VOLUME IS  $|\vec{AE} \cdot (\vec{AB} \wedge \vec{AD})|$ , SO WE OBTAIN

$$\Rightarrow V = |\vec{AE} \cdot (21, 12, 28)|$$

$$\Rightarrow V = |(\underline{e} - \underline{a}) \cdot (21, 12, 28)|$$

$$\Rightarrow V = |[(7, 2, 9) - (5, 1, 3)] \cdot (21, 12, 28)|$$

$$\Rightarrow V = |(2, 1, 6) \cdot (21, 12, 28)|$$

$$\Rightarrow V = |42 + 12 + 168|$$

$$\Rightarrow V = 222$$

As required

c) WE SHOULD OBTAIN THE VOLUME AS

$$\Rightarrow V = \text{BASE AREA} \times \text{HEIGHT}$$

$$\Rightarrow 222 = 37 \times h$$

$$\Rightarrow h = 6$$

IF THE REQUIRED DISTANCE IS 6

1YGB - FP3 PAPER N - QUESTION 7

START BY TRIGONOMETRIC IDENTITIES FIRST

$$\begin{aligned} \lim_{x \rightarrow 0} \left[ \frac{\cos^2 3x - 1}{x^2} \right] &= \lim_{x \rightarrow 0} \left[ \frac{\left( \frac{1}{2} + \frac{1}{2} \cos 6x \right) - 1}{x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[ \frac{\frac{1}{2} \cos 6x - \frac{1}{2}}{x^2} \right] = \lim_{x \rightarrow 0} \left[ \frac{\cos 6x - 1}{2x^2} \right] \end{aligned}$$

USING THE STANDARD EXPANSION OF  $\cos x = 1 - \frac{x^2}{2!} + o(x^4)$

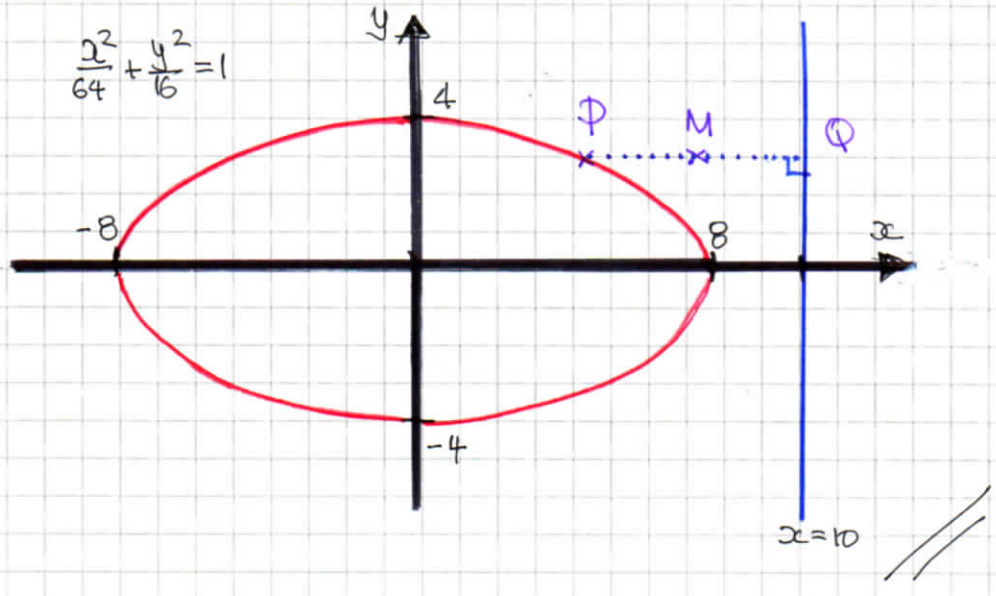
$$\begin{aligned} &= \lim_{x \rightarrow 0} \left[ \frac{\left[ 1 - \frac{(6x)^2}{2!} + o(x^4) \right] - 1}{2x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[ \frac{\cancel{1} - 18x^2 + o(x^4) - \cancel{1}}{2x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[ -9 + o(x^2) \right] \end{aligned}$$

-9

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1YGB - FP3 PAPER N - QUESTION 8

a) THIS IS A STANDARD ELLIPSE WITH  $-8 \leq x \leq 8$   
 $-4 \leq y \leq 4$



b) PARAMETERIZE THE ELLIPSE

$$x = 8\cos\theta, \quad y = 4\sin\theta \quad 0 \leq \theta < 2\pi$$

THEN THE CO-ORDINATES OF P, Q & R CAN BE FOUND

- $P(8\cos\theta, 4\sin\theta)$
- $Q(10, 4\sin\theta)$
- $M\left[\frac{8\cos\theta + 10}{2}, \frac{4\sin\theta + 4\sin\theta}{2}\right] = M(5 + 4\cos\theta, 4\sin\theta)$

ELIMINATE THE PARAMETER  $\theta$ , OUT OF THE GENERAL CO-ORDINATES OF M (WRITTEN AS PARAMETRICS)

$$\left. \begin{array}{l} X = 5 + 4\cos\theta \\ Y = 4\sin\theta \end{array} \right\} \Rightarrow \begin{array}{l} 4\cos\theta = X - 5 \\ 4\sin\theta = Y \end{array}$$

-2-

## 1YGB - FP3 PAPER N - QUESTION 8

$$\Rightarrow \begin{cases} 16\cos^2\theta = (x-5)^2 \\ 16\sin^2\theta = y^2 \end{cases}$$

$$\Rightarrow 16\cos^2\theta + 16\sin^2\theta = (x-5)^2 + y^2$$

$$\Rightarrow 16(\cos^2\theta + \sin^2\theta) = (x-5)^2 + y^2$$

$$\Rightarrow \underline{(x-5)^2 + y^2 = 16}$$

∴ A circle, centered at (5,0), radius 4

1YGB - FP3 PAPER N - QUESTION 9

START BY MANIPULATIONS & AUXILIARIES

●  $t = \tan\left(\frac{1}{2}x\right)$   
 $\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{1}{2}x\right)$   
 $\frac{dt}{dx} = \frac{1}{2} [1 + \tan^2\left(\frac{1}{2}x\right)]$   
 $\frac{dt}{dx} = \frac{1}{2} [1 + t^2]$   
 $\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2} dt$

●  $x = \frac{\pi}{2} \mapsto t = 1$   
 $x = 0 \mapsto t = 0$

●  $t = \tan\frac{1}{2}x$   
 $t^2 = \tan^2\frac{1}{2}x$   
 $1+t^2 = 1 + \tan^2\frac{1}{2}x$   
 $1+t^2 = \sec^2\frac{1}{2}x$   
 $\frac{1}{1+t^2} = \cos^2\frac{1}{2}x$   
 $\frac{2}{1+t^2} = 2\cos^2\frac{1}{2}x$   
 $\frac{2}{1+t^2} - 1 = 2\cos^2\frac{1}{2}x - 1$

$\frac{2 - (1+t^2)}{1+t^2} = \cos\left(2 \times \frac{1}{2}x\right)$

$\frac{1-t^2}{1+t^2} = \cos x$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x} dx = \int_0^1 \frac{1}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{2}{2(1+t^2) - (1-t^2)} dt$$

$$= \int_0^1 \frac{2}{1+3t^2} dt$$

YGB - FP3 PAPER N - QUESTION 9

$$= \frac{2}{3} \int_0^1 \frac{1}{t^2 + \frac{1}{3}} dt$$

MANIPULATE INTO A STANDARD ARCTAN FORM

$$\dots = \frac{2}{3} \int_0^1 \frac{1}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2} dt$$

$$= \frac{2}{3} \times \frac{1}{\frac{1}{\sqrt{3}}} \left[ \arctan\left(\frac{1}{\frac{1}{\sqrt{3}}}t\right) \right]_0^1$$

$$= \frac{2}{3}\sqrt{3} \left[ \arctan(\sqrt{3}t) \right]_0^1$$

$$= \frac{2}{3}\sqrt{3} \left[ \arctan\sqrt{3} - \cancel{\arctan 0} \right]$$

$$= \frac{2}{3}\sqrt{3} \times \frac{\pi}{3}$$

$$= \frac{2\pi\sqrt{3}}{9}$$

