IYGB – FP2 PAPER O – QUESTION 1

$$\boxed{\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 13x^2 - x + 22}$$

SOLVING THE AUXILIARY EQUATION IN THE L.H.S OF THE O.D.E.

$\Rightarrow \lambda^2 + 6\lambda + 13 = 0$

$\Rightarrow (\lambda+3)^2 - 9 + 13 = 0$

$\Rightarrow (\lambda+3)^2 = -4$

$\Rightarrow \lambda+3 = \pm 2i$

$\Rightarrow \lambda = -3 \pm 2i$

COMPLEMENTARY FUNCTION

$y = e^{-3x}(A\cos 2x + B\sin 2x)$

PARTICULAR INTEGRAL BY TRIAL

$$\left.\begin{aligned} y &= Px^2 + Qx + R \\ \frac{dy}{dx} &= 2Px + Q \\ \frac{d^2y}{dx^2} &= 2P \end{aligned}\right\}$$

SUBSTITUTE INTO THE O.D.E & COMPARE

$2P + 6(2Px+Q) + 13(Px^2+Qx+R) \equiv 13x^2 - x + 22$

$13Px^2 + (12P+13Q) + (2P+6Q+13R) \equiv 13x^2 - x + 22$

• $P = 1$	• $12P + 13Q = -1$	• $2P + 6Q + 13R = 22$
	$12 + 13Q = -1$	$2 - 6 + 13R = 22$
	$13Q = -13$	$13R = 26$
	$Q = -1$	$R = 2$

PARTICULAR INTEGRAL IS

$y = x^2 - x + 2$

GENERAL SOLUTION IS

$y = e^{-3x}(A\cos 2x + B\sin 2x) + x^2 - x + 2$

IYGB - FP2 PAPER O - QUESTION 2

a) BY INSPECTION (COVER UP METHOD OR SIMILAR)

$$f(r) = \frac{1}{r(r+2)} = \frac{1/2}{r} + \frac{-\frac{1}{2}}{r+2} = \frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2}$$

$$= \frac{1}{r} - \frac{1}{2(r+2)}$$

b) SETTING PART (a) AS AN IDENTITY

$$\boxed{\frac{2}{r(r+2)} \equiv \frac{1}{r} - \frac{1}{r+2}}$$

- $r=1$ $\qquad \frac{2}{1\times3} = \frac{1}{1} - \cancel{\frac{1}{3}}$
- $r=2$ $\qquad \frac{2}{2\times4} = \frac{1}{2} - \cancel{\frac{1}{4}}$
- $r=3$ $\qquad \frac{2}{3\times5} = \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$
- $r=4$ $\qquad \frac{2}{4\times6} = \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$
- $r=5$ $\qquad \frac{2}{5\times7} = \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

- $r=n-1$ $\qquad \frac{2}{(n-1)(n+1)} = \cancel{\frac{1}{n-1}} - \frac{1}{n+1}$
- $r=n$ $\qquad \frac{2}{n(n+2)} = \cancel{\frac{1}{n}} - \frac{1}{n+2}$

$$\Longrightarrow \sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$ ADDING

$$\Longrightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)}$$

$$\Longrightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3(n^2+3n+2) - 2n - 4 - 2n - 2}{2(n+1)(n+2)}$$

IYGB - FP2 PAPER O - QUESTION 2

$$\Longrightarrow \quad 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3n^2+9n+6-4n-6}{2(n+1)(n+2)}$$

$$\Longrightarrow \quad 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3n^2+5n}{2(n+1)(n+2)}$$

$$\Longrightarrow \quad \sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

IE $A=3$
$B=5$

IYGB – FP2 PAPER O – QUESTION 3

USING STANDARD RESULTS, RATHER THAN DIFFERENTIATION

$$\Longrightarrow f(x) = (1-x)^2 \ln(1-x)$$

$$\Longrightarrow f(x) = (1-2x+x^2)\left[-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + O(x^4)\right]$$

$$\boxed{\begin{aligned} \ln(1+x) &\equiv x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + O(x^5) \\ \ln(1-x) &\equiv -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + O(x^5) \end{aligned}}$$

$$\begin{aligned} \Longrightarrow f(x) = \; & -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + O(x^4) \\ & \qquad\;\; 2x^2 + x^3 + O(x^4) \\ & \qquad\qquad\;\; -x^3 + O(x^4) \end{aligned}$$

$$\Longrightarrow f(x) = -x + \tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 + O(x^4)$$

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MANIPULATE THE SURDS AS FOLLOWS

$$(\sqrt{5}-2)\ln(\sqrt{5}-2) + (\sqrt{5}+2)\ln(\sqrt{5}+2)$$

$$= (\sqrt{5}-2)\ln\left[\frac{(\sqrt{5}-2)(\sqrt{5}+2)}{\sqrt{5}+2}\right] + (\sqrt{5}+2)\ln(\sqrt{5}+2)$$

$$= (\sqrt{5}-2)\ln\left[\frac{1}{\sqrt{5}+2}\right] + (\sqrt{5}+2)\ln(\sqrt{5}+2)$$

$$= -(\sqrt{5}-2)\ln\left[\sqrt{5}+2\right] + (\sqrt{5}+2)\ln\left[\sqrt{5}+2\right]$$

$$= 4\ln\left[2+\sqrt{5}\right]$$

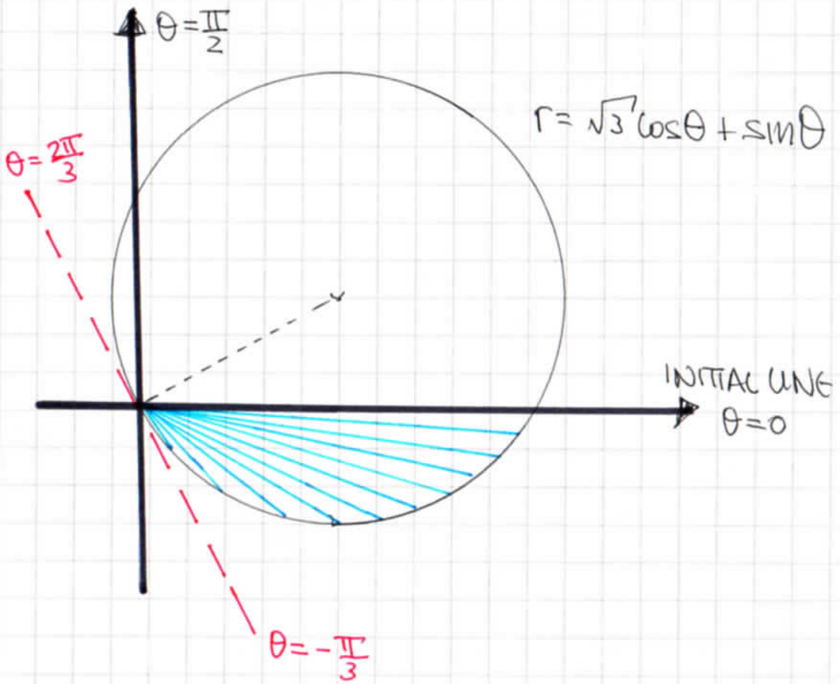
$$= 4\ln\left[2+\sqrt{2^2+1}\right]$$

$$= 4\,\text{arsinh}\,2$$

ie $a=4$
$b=2$

IYGB - FP2 PAPER O - QUESTION 5

LOOKING AT THE DIAGRAM BELOW



$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta = \int_{\theta=-\frac{\pi}{3}}^{\theta=0} \frac{1}{2}\left(\sqrt{3}\cos\theta + \sin\theta\right)^2 d\theta$$

$$\text{AREA} = \int_{-\frac{\pi}{3}}^{0} \frac{1}{2}\left[3\cos^2\theta + \underbrace{2\sqrt{3}\cos\theta\sin\theta} + \sin^2\theta\right] d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \underbrace{2\cos^2\theta + 1} + \sqrt{3}\sin 2\theta \quad d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \overbrace{(1 + \cos 2\theta) + 1} + \sqrt{3}\sin 2\theta \quad d\theta$$

$$\left[\cos 2\theta \equiv 2\cos^2\theta - 1\right]$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} 2 + \cos 2\theta + \sqrt{3}\sin 2\theta \quad d\theta$$

$$\text{AREA} = \frac{1}{2}\left[2\theta + \frac{1}{2}\sin 2\theta - \frac{\sqrt{3}}{2}\cos 2\theta\right]_{-\frac{\pi}{3}}^{0}$$

$$\text{AREA} = \frac{1}{2}\left[\left(0 + 0 - \frac{\sqrt{3}}{2}\right) - \left(-\frac{2\pi}{3} \cancel{- \frac{\sqrt{3}}{4}} \cancel{+ \frac{\sqrt{3}}{4}}\right)\right]$$

$$\text{AREA} = \frac{1}{2}\left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$\text{AREA} = \frac{1}{12}\left[4\pi - 3\sqrt{3}\right]$$

AS REQUIRED

IYGB – FP2 PAPER O – QUESTION 6

REWRITE THE INTEGRAND IN TERMS OF EXPONENTIALS

$$\int_0^{\ln 2} \frac{e^x}{\cosh x}\,dx = \int_0^{\ln 2} \frac{e^x}{\frac{1}{2}(e^x+e^{-x})}\,dx = \int_0^{\ln 2} \frac{2e^x}{e^x+e^{-x}}\,dx$$

NOW BY SUBSTITUTION WE HAVE

$$u = e^x \qquad \& \qquad x=0 \longmapsto u=1$$
$$\frac{du}{dx} = e^x \qquad\qquad x=\ln 2 \longmapsto u=2$$
$$\frac{du}{dx} = u$$
$$dx = \frac{du}{u}$$

TRANSFORMING THE INTEGRAL

$$\ldots = \int_1^2 \frac{2\cancel{u}}{u+u^{-1}}\left(\frac{du}{\cancel{u}}\right) = \int_1^2 \frac{2}{u+\frac{1}{u}}\,du$$

$$= \int_1^2 \frac{2u}{u^2+1}\,du = \Big[\ln(u^2+1)\Big]_1^2$$

$$= \ln 5 - \ln 2 = \ln\frac{5}{2}$$

IYGB - FP2 PAPER O - QUESTION 7

a) LET $\cos\theta + i\sin\theta \equiv c + is$, AND RAISE BOTH SIDES OF THIS EXPRESSION TO THE POWER OF 5

$$\Rightarrow (\cos\theta + i\sin\theta)^5 = (c+is)^5$$

$$\Rightarrow \cos 5\theta + i\sin 5\theta = (c+is)^5$$

FOLLOWING THE PATTERN

$+ \quad + \quad - \quad - \quad + \quad + \quad \dots$

Re Im Re Im Re Im ...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\Rightarrow \cos 5\theta + i\sin 5\theta = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$$

$$\Rightarrow \cos 5\theta + i\sin 5\theta = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$$

$$\Rightarrow \sin 5\theta = 5(1-s^2)^2 s - 10(1-s^2)s^3 + s^5$$

$$\Rightarrow \sin 5\theta = 5s(s^4 - 2s^2 + 1) - 10s^3 + 10s^5 + s^5$$

$$\Rightarrow \sin 5\theta = 5s^5 - 10s^3 + 5s - 10s^3 + 10s^5 + s^5$$

$$\Rightarrow \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

AS REQUIRED

b) START BY SOLVING THE EQUATION $\sin 5\theta = 0$

- $\sin 5\theta = 0$

$5\theta = n\pi \qquad n \in \mathbb{Z}$

$\theta = \frac{n\pi}{5}$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \dots$

IYGB - FP2 PAPER O - QUESTION 7

ALSO BY LETTING $x = \sin\theta$, THE R.H.S YIELDS

$$x(16x^4 - 20x^2 + 5) = 0$$

$$\sin\theta(16\sin^4\theta - 20\sin^2\theta + 5) = 0$$

- $\theta = 0$ IS FROM THE FACTORIZED $\sin\theta$ (OR $\theta = \pi$)
- $x = \sin\frac{\pi}{5}$, $x = \sin\frac{2\pi}{5}$, $x = \sin\frac{6\pi}{5}$, $x = \sin\frac{7\pi}{5}$

  OR $\left(\sin\frac{4\pi}{5}\right)$ $\left(\sin\frac{3\pi}{5}\right)$ $\left(x = \sin\frac{9\pi}{5}\right)$ $\left(x = \sin\frac{8\pi}{5}\right)$

c) SOLVING THE QUARTIC BY THE QUADRATIC FORMULA

- $16x^4 - 20x^2 + 5 = 0 \implies x^2 = \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32}$

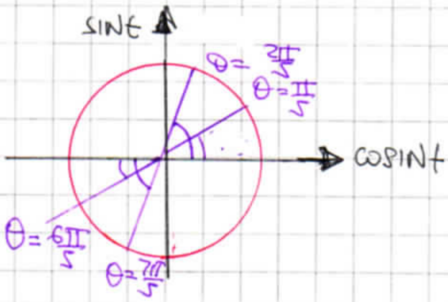
  $\implies x^2 = \frac{5 \pm \sqrt{5}}{8}$

- $\left(x - \sin\frac{\pi}{5}\right)\left(x - \sin\frac{6\pi}{5}\right)\left(x - \sin\frac{2\pi}{5}\right)\left(x - \sin\frac{7\pi}{5}\right) = 0$

  $\left(x - \sin\frac{\pi}{5}\right)\left(x - \sin\left(-\frac{\pi}{5}\right)\right)\left(x - \sin\frac{2\pi}{5}\right)\left(x - \sin\left(-\frac{2\pi}{5}\right)\right) = 0$

  $\left(x - \sin\frac{\pi}{5}\right)\left(x + \sin\frac{\pi}{5}\right)\left(x - \sin\frac{2\pi}{5}\right)\left(x + \sin\frac{2\pi}{5}\right) = 0$

  $\left(x^2 - \sin^2\frac{\pi}{5}\right)\left(x^2 - \sin^2\frac{2\pi}{5}\right) = 0$



- $\sin^2\frac{\pi}{5} = \begin{cases} \frac{5 + \sqrt{5}}{8} \\ \text{OR} \\ \frac{5 - \sqrt{5}}{8} \end{cases}$

  BUT $\sin\frac{\pi}{6} < \sin\frac{\pi}{5} < \sin\frac{\pi}{4}$

  $\sin^2\frac{\pi}{6} < \sin^2\frac{\pi}{5} < \sin^2\frac{\pi}{4}$

  $\frac{1}{4} < \sin^2\frac{\pi}{6} < \frac{1}{2}$

  $\therefore \sin^2\frac{\pi}{5} \neq \frac{5}{8} + \frac{\sqrt{5}}{8} > \frac{1}{2}$

$$\sin^2\frac{\pi}{5} = \frac{5}{8} - \frac{1}{8}\sqrt{5}$$

IYGB - FP2 PAPER O - QUESTION 8

a) PROCEED AS "ADVISED"

$$\Rightarrow y = \arccos x$$
$$\Rightarrow \cos y = x$$
$$\Rightarrow x = \cos y$$
$$\Rightarrow \frac{dx}{dy} = -\sin y$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{+\sqrt{1-\cos^2 y}}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

// AS REQUIRED

$\sin^2 y + \cos^2 y = 1$

$\sin y = \pm\sqrt{1-\cos^2 y}$

$0 \leqslant y \leqslant \pi$, SO THAT $\sin y$ CANNOT BE A NEGATIVE QUANTITY

b) DIFFERENTIATING THE EQUATION OF THE CURVE

$$\Rightarrow y = \arccos x - \tfrac{1}{2}\ln(1-x^2)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} - \frac{\cancel{1}}{\cancel{2}} \times \frac{1}{1-x^2} \times (\cancel{-2}x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{1-x^2} - \frac{1}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x-\sqrt{1-x^2}}{1-x^2}$$

SOLVING FOR ZERO YIELDS

$$\Rightarrow x - \sqrt{1-x^2} = 0$$
$$\Rightarrow x = \sqrt{1-x^2}$$
$$\Rightarrow x^2 = 1-x^2$$

IYGB - FP2 PAPER O - QUESTION 8

$\Longrightarrow\ 2x^2 = 1$

$\Longrightarrow\ x^2 = \frac{1}{2}$

$\Longrightarrow\ x = \pm\frac{1}{\sqrt{2}}$

$\Longrightarrow\ x = +\frac{1}{\sqrt{2}}$ AS $x = -\frac{1}{\sqrt{2}}$ DOES NOT SATISFY THE EQUATION $x = \sqrt{1-x^2}$ — THIS EXTRA SOLUTION IS DUE TO SQUARING

FINDING THE y CO-ORDINATE

$\Longrightarrow\ y = \arccos x - \frac{1}{2}\ln(1-x^2)$

$\Longrightarrow\ y = \arccos\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\ln\left(1-\frac{1}{2}\right)$

$\Longrightarrow\ y = \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}$

$\Longrightarrow\ y = \frac{\pi}{4} + \frac{1}{2}\ln 2$

$\Longrightarrow\ y = \frac{1}{4}\left[\pi + 2\ln 2\right]$

$\Longrightarrow\ y = \frac{1}{4}(\pi + \ln 4)$

// AS REQUIRED

IYGB - FP2 PAPER O - QUESTION 9

$$\boxed{(1-x^2)\frac{dy}{dx} + y = (1-x^2)(1-x)^{\frac{1}{2}}}$$

REWRITE THE O.D.E IN "STANDARD" FORM AND LOOK FOR AN INTEGRATING FACTOR

$$\Longrightarrow \frac{dy}{dx} + \frac{1}{1-x^2}\frac{dy}{dx} = (1-x)^{\frac{1}{2}}$$

$$\text{I.F.} = e^{\int \frac{1}{1-x^2}\,dx} = e^{\int \frac{1}{(1+x)(1-x)}\,dx} = \dots\dots$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= e^{\int \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}\,dx} = e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|} = e^{\ln\sqrt{\frac{1+x}{1-x}}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$\Longrightarrow \frac{d}{dx}\left[y\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)\right] = \cancel{(1-x)^{\frac{1}{2}}}\left(\frac{\sqrt{1+x}}{\cancel{\sqrt{1-x}}}\right)$$

$$\Longrightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = \int (1+x)^{\frac{1}{2}}\,dx$$

$$\Longrightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + A$$

$$\Longrightarrow \boxed{y = \frac{2}{3}(1+x)^{1}(1-x)^{\frac{1}{2}} + A\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}}$$

APPLY $x = \frac{1}{2}$, $y = \frac{\sqrt{2}}{2}$

$$\Longrightarrow \frac{\sqrt{2}}{2} = \frac{2}{3}\times\frac{3}{2}\times\frac{\sqrt{2}}{2} + A\frac{\sqrt{2}/2}{3/2}$$

$$\Longrightarrow \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + A\frac{\sqrt{2}}{3}$$

$$\Longrightarrow A = 0$$

IYGB - FP2 PAPER O - QUESTION 9

$$\Rightarrow \quad y = \frac{2}{3}(1+x)(1-x)^{\frac{1}{2}}$$

$$\Rightarrow \quad y = \frac{2}{3}(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$$

$$\Rightarrow \quad y = \frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{(1+x)(1-x)}$$

$$\Rightarrow \quad y = \frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{1-x^2}$$

$$\Rightarrow \quad y = \frac{2}{3}\sqrt{(1+x)(1-x^2)}$$

AS REQUIRED