

1 YGB - FP2 PAPER N - QUESTION 1

a) OBTAIN A COMMON DENOMINATOR AND ADD

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$

b) USING THE IDENTITY PROVEN IN PART (a)

$$\frac{2}{r(r+1)(r+2)} \equiv \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

- IF  $r=1$  :  $\frac{2}{1 \times 2 \times 3} = \frac{1}{1 \times 2} - \frac{1}{2 \times 3}$
- IF  $r=2$  :  $\frac{2}{2 \times 3 \times 4} = \frac{1}{2 \times 3} - \frac{1}{3 \times 4}$
- IF  $r=3$  :  $\frac{2}{3 \times 4 \times 5} = \frac{1}{3 \times 4} - \frac{1}{4 \times 5}$
- $\vdots$
- IF  $r=20$  :  $\frac{2}{20 \times 21 \times 22} = \frac{1}{20 \times 21} - \frac{1}{21 \times 22}$

$$\Rightarrow \sum_{r=1}^{20} \left[ \frac{2}{r(r+1)(r+2)} \right] = \frac{1}{1 \times 2} - \frac{1}{21 \times 22}$$

ADDING

$$\Rightarrow 2 \sum_{r=1}^{20} \left[ \frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} - \frac{1}{462}$$

$$\Rightarrow \sum_{r=1}^{20} \left[ \frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{462} \right) = \frac{115}{462}$$

AS REQUIRED

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1YGB - FP2 PAPER N - QUESTION 2

USING STANDARD EXPANSIONS

●  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^5)$

$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + o(x^5)$

$e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^5)$

●  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^6)$

$\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + o(x^6)$

$\cos 4x = 1 - 8x^2 + \frac{32}{3}x^4 + o(x^6)$

COMBINING THESE RESULTS

$f(x) = e^{-2x} \cos 4x = (\cos 4x)(e^{-2x})$

$f(x) = \left[ 1 - 8x^2 + \frac{32}{3}x^4 + o(x^6) \right] \left[ 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^5) \right]$

$f(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^5)$   
 $\quad - 8x^2 + 16x^3 - 16x^4 + o(x^5)$   
 $\quad + \frac{32}{3}x^4 + o(x^5)$

$f(x) = 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + o(x^5)$

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## IYGB - FP2 PAPER N - QUESTION 3

a) FACTORIZE THE DENOMINATOR FIRST

$$f(y) = \frac{4y}{y^3-1} = \frac{4y}{(y^2-1)(y+1)} = \frac{4y}{(y-1)(y+1)(y^2+1)}$$

NOW WE HAVE

$$\frac{4y}{(y-1)(y+1)(y^2+1)} \equiv \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cx+D}{y^2+1}$$

$$\Rightarrow \boxed{4y \equiv A(y+1)(y^2+1) + B(y-1)(y^2+1) + (y-1)(y+1)(Cx+D)}$$

• If  $y=1$

$$4 = 4A$$

$$\underline{A=1}$$

• If  $y=-1$

$$-4 = -4B$$

$$\underline{B=1}$$

• If  $y=0$

$$0 = A - B - D$$

$$\underline{D=0}$$

• If  $y=2$

$$8 = 15A + 5B + 6C$$

$$8 = 15 + 5 + 6C$$

$$-12 = 6C$$

$$\underline{C=-2}$$

$$\therefore \underline{f(y) = \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}}$$

b)  $\int_2^{\infty} f(y) dy = \int_2^{\infty} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1} dy$

$$= \lim_{a \rightarrow \infty} \left[ \int_2^a \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1} dy \right]$$

$$= \lim_{a \rightarrow \infty} \left[ \left[ \ln(y-1) + \ln(y+1) - \ln(y^2+1) \right]_2^a \right]$$

$$= \lim_{a \rightarrow \infty} \left[ \left[ \ln \left( \frac{y^2-1}{y^2+1} \right) \right]_2^a \right]$$

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$$= \lim_{a \rightarrow \infty} \left[ \ln \left( \frac{a^2-1}{a^2+1} \right) - \ln \left( \frac{3}{5} \right) \right]$$

$$= \ln 1 - \ln \frac{3}{5}$$

$$= \ln \frac{5}{3}$$

c) USING THE INTEGRATION OF PART (b) WE HAVE

$$\text{MEAN OF } f(y), \text{ OVER } [2, 4] = \frac{1}{4-2} \int_2^4 f(y) dy$$

$$= \frac{1}{2} \left[ \ln \left( \frac{a^2-1}{a^2+1} \right) - \ln \left( \frac{3}{5} \right) \right]$$

WHERE  $a=4$

$$= \frac{1}{2} \left[ \ln \frac{15}{17} - \ln \frac{3}{5} \right]$$

$$= \frac{1}{2} \left[ \ln \frac{15}{17} + \ln \frac{5}{3} \right]$$

$$= \frac{1}{2} \ln \frac{25}{17}$$

OR  $\ln \left( \frac{5}{\sqrt{17}} \right)$

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NYGB - FP2 PAPER 1 - QUESTION 4

a) WORKING IN EXPONENTIALS

$$\begin{aligned} z = e^{i\theta} &\Rightarrow z^n = (e^{i\theta})^n \\ &\Rightarrow z^n = e^{in\theta} \\ &\Rightarrow z^{-n} = e^{-in\theta} \end{aligned}$$

Hence we have

$$\begin{aligned} \text{I) } z^n + \frac{1}{z^n} &= z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = 2\cosh(in\theta) \\ &= \underline{2\cos n\theta} \end{aligned}$$

$$\begin{aligned} \text{II) } z^n - \frac{1}{z^n} &= z^n - z^{-n} = e^{in\theta} - e^{-in\theta} = 2\sinh(in\theta) \\ &= \underline{2i\sin n\theta} \end{aligned}$$

OR USING TRIGONOMETRIC FUNCTIONS VIA Euler's formula

$$\begin{aligned} z^n + \frac{1}{z^n} &= e^{in\theta} + e^{-in\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) \\ &= 2\cos n\theta \end{aligned}$$

& SIMILARLY THE OTHER

b) START BY NOTING THAT IF  $n=1$

$$z + \frac{1}{z} = 2\cos\theta \quad \& \quad z - \frac{1}{z} = 2i\sin\theta$$

SUBSTITUTE & EXPAND BINOMIALLY

$$\Rightarrow \left(z + \frac{1}{z}\right)^4 \left(z - \frac{1}{z}\right)^2 = (2\cos\theta)^4 (2i\sin\theta)^2$$

1YGB - FP2 PAPER N - QUESTION 4

$$\Rightarrow (6\cos^4\theta)(-4\sin^2\theta) = \left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^4$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^2 - \frac{1}{z^2}\right)^2 \left(z + \frac{1}{z}\right)^2$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^4 - 2 + \frac{1}{z^4}\right) \left(z^2 + 2 + \frac{1}{z^2}\right)$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 + z^2 - 2z^2 - 4 - \frac{2}{z^2} + \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 - z^2 - 4 - \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^6 + \frac{1}{z^6}\right) + 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) - 4$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = (2\cos 6\theta) + 2(2\cos 4\theta) - (2\cos 2\theta) - 4$$

$$\Rightarrow -64\cos^4\theta\sin^2\theta = -4 - 2\cos 2\theta + 4\cos 4\theta + 2\cos 6\theta$$

$$\Rightarrow \cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$$

AS REQUIRED

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1YGB - FP2 PAPER N - QUESTION 5

$$f(x) = e^{2x+2}(e^{2x} - 4), x \in \mathbb{R}$$

If  $x = \ln(2\cosh \frac{1}{2})$

$$e^{2x} = e^{2\ln(2\cosh \frac{1}{2})} = e^{\ln(2\cosh \frac{1}{2})^2} = e^{\ln(4\cosh^2 \frac{1}{2})} = 4\cosh^2 \frac{1}{2}$$

$$e^{2x+2} = e^2(4\cosh^2 \frac{1}{2}) = 4e^2\cosh^2 \frac{1}{2}$$

Hence we have

$$\begin{aligned} f(\ln(2\cosh \frac{1}{2})) &= 4e^2\cosh^2 \frac{1}{2} (4\cosh^2 \frac{1}{2} - 4) \\ &= 16e^2\cosh^2 \frac{1}{2} (\cosh^2 \frac{1}{2} - 1) \\ &= 16e^2\cosh^2 \frac{1}{2} (\sinh^2 \frac{1}{2}) \quad \left. \begin{array}{l} \text{cosh}^2 A - \sinh^2 A \equiv 1 \end{array} \right\} \\ &= 4e^2 (4\sinh^2 \frac{1}{2} \cosh^2 \frac{1}{2}) \\ &= 4e^2 (2\sinh \frac{1}{2} \cosh \frac{1}{2})^2 \quad \left. \begin{array}{l} \sinh 2A \equiv 2\sinh A \cosh A \end{array} \right\} \\ &= 4e^2 (\sinh(2 \times \frac{1}{2}))^2 \\ &= 4e^2 \sinh^2 1 \\ &= (2e\sinh 1)^2 \\ &= [2e \times \frac{1}{2} (e^1 - e^{-1})]^2 \\ &= (e^2 - 1)^2 \end{aligned}$$

1Y0B - FP2 PAPER N - QUESTION 6

a) CONDITION FOR A "HORIZONTAL TANGENT"

$$\frac{dy}{dx} = 0 \Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0$$

$$\Rightarrow dy/d\theta = 0$$

$$\Rightarrow \frac{d}{d\theta}(y) = 0$$

$$\Rightarrow \frac{d}{d\theta}(r \sin \theta)$$

$$\Rightarrow \frac{d}{d\theta}[4(1 - \sin \theta) \sin \theta] = 0$$

$$\Rightarrow \frac{d}{d\theta}[\sin \theta - \sin^2 \theta] = 0$$

$$\Rightarrow \cos \theta - 2 \sin \theta \cos \theta = 0$$

$$\Rightarrow \cos \theta (1 - 2 \sin \theta) = 0$$

HENCE THE SOLUTIONS FOR  $0 \leq \theta \leq \pi$

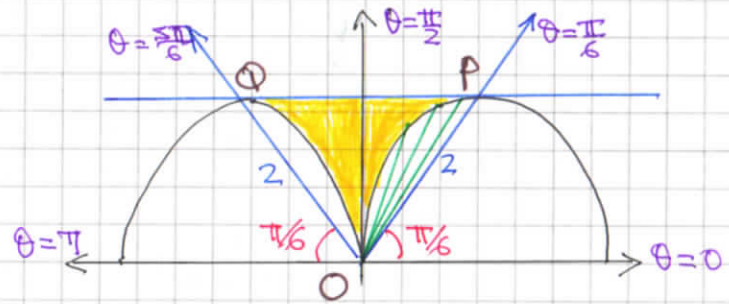
$$\cos \theta = 0 \Rightarrow \theta = \pi/2$$

$$\sin \theta = \frac{1}{2} \Rightarrow \theta = \left\langle \begin{matrix} \pi/6 \\ 5\pi/6 \end{matrix} \right.$$

$$\Rightarrow r = \left\langle \begin{matrix} 4(1 - \frac{1}{2}) = 2 \\ 4(1 - \frac{1}{2}) = 2 \end{matrix} \right.$$

$$\therefore \underline{P(2, \frac{\pi}{6})} \text{ \& } \underline{Q(2, \frac{5\pi}{6})}$$

b) LOOKING AT THE DIAGRAM BELOW



$$\begin{aligned} \underline{\text{AREA OF } \triangle OPQ} &= \frac{1}{2} |OP| |OQ| \sin(\pi - 2 \times \frac{\pi}{6}) \\ &= \frac{1}{2} \times 2 \times 2 \times \sin(\frac{2\pi}{3}) \\ &= \sqrt{3} \end{aligned}$$



1YGB - FP2 PAPER N - QUESTION 6

AREA OF THE "GREEN" POLAR SECTORS FROM  $\theta = \frac{\pi}{6}$  TO  $\theta = \frac{\pi}{2}$

$$\begin{aligned}
 A &= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} \frac{1}{2} [4(1-\sin\theta)]^2 d\theta = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} 4(1-\sin\theta)^2 d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4(1-2\sin\theta + \sin^2\theta) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 8\sin\theta + 4\sin^2\theta d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 8\sin\theta + 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 6 - 8\sin\theta - 2\cos 2\theta d\theta = \left[ 6\theta + 8\cos\theta - \sin 2\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= (6\pi + 0 - 0) - (2\pi + 8\sqrt{3} - \sqrt{3}) = 4\pi - 7\sqrt{3}
 \end{aligned}$$

THUS THE REQUIRED AREA IS GIVEN BY

$$\begin{aligned}
 &\sqrt{3} - 2(4\pi - 7\sqrt{3}) \\
 &= \sqrt{3} - 8\pi + 14\sqrt{3} \\
 &= \underline{15\sqrt{3} - 8\pi} \quad \text{AS REQUIRED}
 \end{aligned}$$

1YGB - FP2 PAPER 1 - QUESTION 7

START BY WRITING THE R.H.S OF THE EQUATION IN EXPONENTIAL FORM

$$|1 + i\sqrt{3}| = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = 2$$

$$\arg(1 + i\sqrt{3}) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$$

$$|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\arg(1 - i) = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$$

REWRITING THE EQUATION

$$z^3 = (1 + i\sqrt{3})^8 (1 - i)^5$$

$$z^3 = [2e^{i\frac{\pi}{3}}]^8 [\sqrt{2}e^{-i\frac{\pi}{4}}]^5$$

(IGNORE MULTIPLES OF  $2\pi$  AT THIS STAGE)

$$z^3 = 2^6 e^{i\frac{8\pi}{3}} \times 4\sqrt{2} e^{-i\frac{5\pi}{4}}$$

$$z^3 = 2^8 \times 2^2 \times 2^{\frac{1}{2}} \times e^{i\frac{17\pi}{12}}$$

$$z^3 = 2^{\frac{21}{2}} e^{i\frac{17\pi}{12} + 2k\pi}, k \in \mathbb{Z}$$

(INTRODUCE MULTIPLES OF  $2\pi$ )

$$(z^3)^{\frac{1}{3}} = [2^{\frac{21}{2}} e^{i\frac{\pi}{12}(17+24k)}]^{\frac{1}{3}}$$

$$z = 2^{\frac{7}{2}} e^{i\frac{\pi}{36}(24k+17)}$$

WORKING THE SOLUTIONS FOR  $-\pi < \theta \leq \pi$

$$z_0 = 8\sqrt{2} e^{i\frac{17\pi}{36}}$$

$$z_{-1} = 8\sqrt{2} e^{-i\frac{7\pi}{36}}$$

$$z_{-2} = 8\sqrt{2} e^{i\frac{31\pi}{36}}$$

# IYGB - FP2 PAPER N - QUESTION 8

$$\frac{dx}{dt} = x + \frac{2}{3}y \quad \frac{dy}{dt} = 3y - \frac{3}{2}x \quad \begin{matrix} t=0 \\ x=1 \\ y=3 \end{matrix}$$

DIFFERENTIATE THE FIRST O.D.E WITH RESPECT TO t

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3} \frac{dy}{dt}} \quad \text{- I}$$

SUBSTITUTE THE SECOND O.D.E. INTO THE ABOVE EXPRESSION

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3} \left( 3y - \frac{3}{2}x \right)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2y - x} \quad \text{- II}$$

REARRANGE THE FIRST O.D.E

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow 3 \frac{dx}{dt} = 3x + 2y$$

$$\Rightarrow \boxed{2y = 3 \frac{dx}{dt} - 3x} \quad \text{- III}$$

COMBINING (II) & (III) WE OBTAIN

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \left( 3 \frac{dx}{dt} - 3x \right) + x$$

1YGB - FP2 PAPER N - QUESTION 8

$$\Rightarrow \frac{d^2x}{dt^2} = 4 \frac{dx}{dt} - 4x$$

$$\Rightarrow \frac{d^2x}{dt^2} - 4 \frac{dx}{dt} + 4x = 0$$

AUXILIARY EQUATION FOR THE ABOVE O.D.E

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2 \text{ (REPEATS)}$$

GENERAL SOLUTION FOR  $x = f(t)$

$$\Rightarrow x = f(t) = Ae^{2t} + Bte^{2t}$$

$$\Rightarrow x = f(t) = e^{2t}(A + Bt)$$

APPLY CONDITION  $t=0, x=1$  YIELDS  $A=1$

$$\Rightarrow \boxed{x = f(t) = e^{2t}(1 + Bt)}$$

NOW DIFFERENTIATE  $x$  & SUB INTO THE FIRST O.D.E

$$\begin{aligned} \Rightarrow \frac{dx}{dt} &= 2e^{2t}(1 + Bt) + Be^{2t} \\ &= e^{2t}(2 + B + 2Bt) \end{aligned}$$

$$\Rightarrow e^{2t}(2 + B + 2Bt) = e^{2t}(1 + Bt) + \frac{2}{3}y$$

$$\frac{dx}{dt} = x + \frac{2}{3}y$$

1YGB - FP2 PAPER N - QUESTION 8

$$\Rightarrow \frac{2}{3}y = e^{2t}(2+B+2Bt) - e^{2t}(1+Bt)$$

$$\Rightarrow \frac{2}{3}y = e^{2t}(1+B+Bt)$$

$$\Rightarrow \boxed{y = \frac{3}{2}e^{2t}(1+B+Bt)}$$

FINALLY APPLY THE CONDITION,  $t=0$   $y=3$

$$\Rightarrow 3 = \frac{3}{2}(1+B)$$

$$\Rightarrow 2 = B+1$$

$$\Rightarrow \underline{B=1}$$

$$x = f(t) = e^{2t}(1+t)$$

$$y = g(t) = \frac{3}{2}e^{2t}(2+t)$$