C4, IYGB, PAPER X

1. $e^{y} = x^{x}$

$\Rightarrow \ln(ye^{y}) = \ln(x^{x})$

$\Rightarrow \ln y + \ln e^{y} = x\ln x$

$\Rightarrow \ln y + y = x\ln x$

Diff w.r.t x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} + 1\frac{dy}{dx} = 1\times\ln x + x\times\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}+1\right) = \ln x + 1$

$\Rightarrow \frac{dy}{dx} = \frac{1+\ln x}{\frac{1}{y}+1}$ MULTIPLY TOP & BOTTOM BY y

$\Rightarrow \frac{dy}{dx} = \frac{y(1+\ln x)}{1+y}$ // AS REQUIRED

2. a) $(3+2x)^{n} = 3^{n}\left(1+\frac{2}{3}x\right)^{n}$

$= 3^{n}\left[1 + \frac{n}{1}\left(\frac{2}{3}x\right) + \frac{n(n-1)}{1\times2}\left(\frac{2}{3}x\right)^{2} + \frac{n(n-1)(n-2)}{1\times2\times3}\left(\frac{2}{3}x\right)^{3} + O(x^{4})\right]$

$= 3^{n}\left[1 + \frac{2}{3}nx + \frac{2}{9}n(n-1)x^{2} + \frac{4}{81}n(n-1)(n-2)x^{3} + O(x^{4})\right]$

THUS $\frac{\frac{4}{81}\cancel{n}\cancel{(n-1)}(n-2)}{\frac{2}{9}\cancel{n}\cancel{(n-1)}} = \frac{2(n-2)}{9}$ IF $2(n-2) : 9$ //

b) $2\left(\frac{7}{2}-2\right) : 9$

$3 : 9$

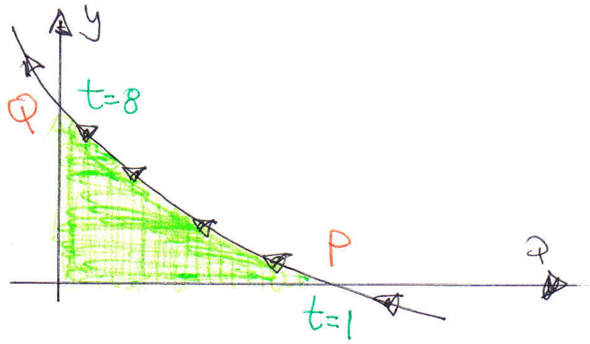
$1 : 3$ //

c) $$\frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)}{1\times2\times3\times4\times5}\left(\frac{2}{3}x\right)^5$$

(FIRST NEGATIVE COEFFICIENT)

$\therefore r=5$

3. a)



$$x = 2-\frac{1}{4}t$$
$$y = 2^t-2$$

• P: $y=0$

$2^t-2=0$

$2^t=2$

$t=1$

$\therefore x = 2-\frac{1}{4}\times1=\frac{7}{4}$

$P\left(\frac{7}{4},0\right)$

• Q: $x=0$

$2-\frac{1}{4}t=0$

$2=\frac{1}{4}t$

$t=8$

$y=2^8-2=254$

$Q(0,254)$

b) CURVE IS TRACED "BACKWARDS" (SEE ARROWS & VALUES OF t)

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$$

$$= \int_{t=8}^{t=1}(2^t-2)\left(-\frac{1}{4}\,dt\right) = \int_1^8 \frac{1}{4}(2^t-2)\,dt$$

$$= \int_1^8 \frac{1}{4}\times2^t-\frac{1}{2}\,dt = \int_1^8 2^{-2}\times2^t-\frac{1}{2}\,dt$$

$$= \int_1^8 2^{t-2} - \frac{1}{2}\, dt$$ AS REQUIRED

c) INTEGRATING

$$\ldots = \left[\frac{1}{\ln 2}2^{t-2} - \frac{1}{2}t\right]_1^8$$

NOTE
$$\frac{d}{dx}(a^x) = a^x \ln a$$
THUS
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C$$

$$= \left(\frac{1}{\ln 2}\times 64 - 4\right) - \left(\frac{1}{\ln 2}\times\frac{1}{2} - \frac{1}{2}\right)$$

$$= \frac{127}{2\ln 2} - \frac{7}{2}$$

4. a) $f(x) = \frac{1}{8}(4x + \sin 4x)$

$$f'(x) = \frac{1}{8}(4 + 4\cos 4x) = \frac{1}{2}(1 + \cos 4x)$$

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 4A = 2\cos^2 2A - 1$$

$$= \frac{1}{2}\left[\cancel{1} + (2\cos^2 2x - \cancel{1})\right] = \frac{1}{2}\times 2\cos^2 2x = \cos^2 2x$$ AS REQUIRED

b) $$V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_0^{\frac{\pi}{4}} (\sqrt{x}\cos 2x)^2\,dx$$

$$V = \pi\int_0^{\frac{\pi}{4}} x\cos^2 2x\,dx$$ BY PARTS, IGNORING π & LIMITS

HENCE $\int x\cos^2 2x \, dx$

x	1
$\frac{1}{8}(4x+\sin 4x)$	$\cos^2 2x$

$$= \frac{1}{8}x(4x+\sin 4x) - \int \frac{1}{8}(4x+\sin 4x)\,dx$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \int \frac{1}{2}x + \frac{1}{8}\sin 4x \, dx$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \left[\frac{1}{4}x^2 - \frac{1}{32}\cos 4x\right] + C$$

$$= \frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C$$

$$\therefore V = \pi\left[\frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x\right]_0^{\frac{\pi}{4}}$$

$$V = \pi\left[\left(\frac{\pi^2}{64} + 0 - \frac{1}{32}\right) - \left(0 + 0 + \frac{1}{32}\right)\right]$$

$$V = \pi\left[\frac{\pi^2}{64} - \frac{1}{16}\right]$$

$$V = \frac{\pi}{64}(\pi^2 - 4)$$

(P.T.O)

5. a)

$$\underline{r}_1 = (2,1,5) + \lambda(1,0,-1) = (\lambda+2, 1, 5-\lambda)$$
$$\underline{r}_2 = (2,1,5) + \mu(1,4,-1) = (\mu+2, 4\mu+1, 5-\mu)$$

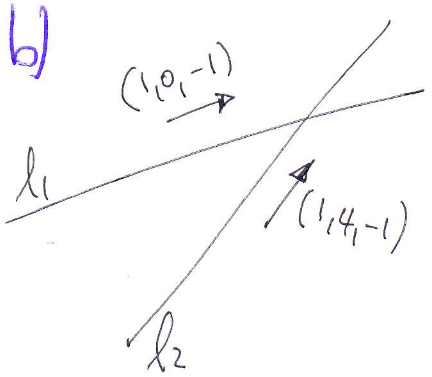
- $\underline{b} = (b,1,-1) = (\lambda+2, 1, 5-\lambda)$

$$5-\lambda = -1$$
$$\boxed{6 = \lambda}$$
$$\therefore \; b = \lambda+2$$
$$b = 8$$

- $\underline{d} = (4,d,3) = (\mu+2, 4\mu+1, 5-\mu)$

$$\mu+2 = 4$$
$$\boxed{\mu = 2}$$
$$\therefore \; d = 4\mu+1$$
$$d = 9$$

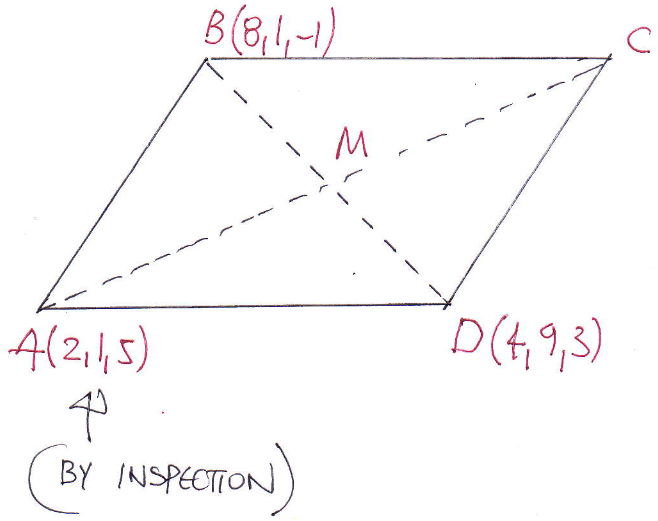
b)



DOTTING THE DIRECTION VECTORS OF THE TWO LINES

$$(1,0,-1)\cdot(1,4,-1) = |1,0,-1||1,4,-1|\cos\theta$$
$$1+0+1 = \sqrt{1+0+1}\sqrt{1+16+1}\cos\theta$$
$$2 = \sqrt{2}\sqrt{18}\cos\theta$$
$$2 = 6\cos\theta$$
$$\cos\theta = \frac{1}{3}$$

c)



- MIDPOINT OF BD IS $M(6,5,1)$
- M IS ALSO THE MIDPOINT OF AC

$$A(2,1,5) \quad M(6,5,1) \quad C(10,9,-3)$$

$+4$ $+4$ -4 $+4$ $+4$

$$\therefore \; C(10,9,-3)$$

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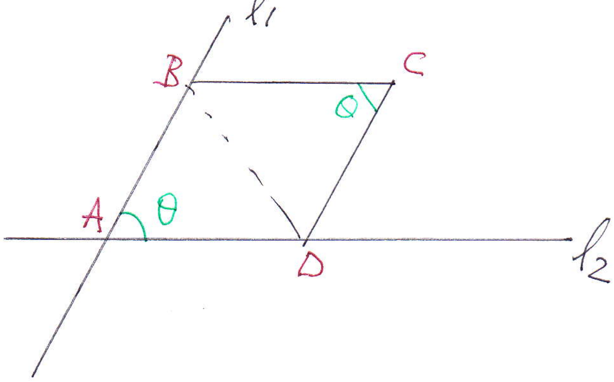
d)



$\cos\theta = \frac{1}{3}$

- $|\vec{AB}| = |\underline{b} - \underline{a}| = |(8,1,-1) - (2,1,5)| = |6,0,-6| = \sqrt{36+36}$

  $= \sqrt{72} = 6\sqrt{2}$

- $|\vec{AD}| = |\underline{d} - \underline{a}| = |(4,9,3) - (2,1,5)| = |2,8,-2| = \sqrt{4+64+4}$

  $= \sqrt{72} = 6\sqrt{2}$

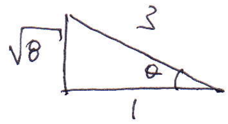
$\cos\theta = \frac{1}{3} \implies \sin\theta = +\sqrt{1 - \cos^2\theta}$

$\sin\theta = +\sqrt{1 - \left(\frac{1}{3}\right)^2}$

$\sin\theta = \sqrt{\frac{8}{9}}$

$\boxed{\sin\theta = \frac{2}{3}\sqrt{2}}$

OR



$\therefore \sin\theta = \frac{\sqrt{8}}{3}$

$\therefore$ AREA OF PARALLELOGRAM $=$ (TRIANGLE ABD) $\times 2$

$= \frac{1}{2}|\vec{AB}||\vec{AD}|\sin\theta \times 2$

$= \cancel{\frac{1}{2}}\sqrt{72}\sqrt{72} \times \frac{2}{3}\sqrt{2} \times \cancel{2}$

$= 48\sqrt{2}$

6.

$$\left.\frac{dA}{dt}\right|_{r=12} = -6 \quad \text{(GIVEN)}$$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dr}{dA} = \frac{1}{8\pi r}$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \left[\frac{dV}{dr} \times \frac{dr}{dA}\right] \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{1}{8\pi r} \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{2}r \times \frac{dA}{dt}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=12} = \frac{1}{2} \times 12 \times \left.\frac{dA}{dt}\right|_{r=12}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=12} = \frac{1}{2} \times 12 \times (-6) = -36 \text{ cm}^3\text{s}^{-1}$$

7. a)

x = infected chickens (1000s)

t = time (hours)

$t = 0, \ x = 4$

$, \ \frac{dx}{dt} = 0.032$

$$\frac{dx}{dt} = kx(20-x)$$

infected   not infected

b) APPLY CONDITION $x = 4, \frac{dx}{dt} = 0.032$

$0.032 = k \times 4 \times 16$

$64k = 0.032$

$k = \frac{1}{2000}$

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Thus $\frac{dx}{dt} = \frac{1}{2000}x(20-x)$, SEPARATE VARIABLES

$$\Rightarrow \frac{2000}{x(20-x)}\,dx = 1\,dt$$

$$\Rightarrow \int \frac{2000}{x(20-x)}\,dx = \int 1\,dt$$

BY PARTIAL FRACTIONS

$$\frac{2000}{x(20-x)} \equiv \frac{A}{x} + \frac{B}{20-x}$$

$$2000 \equiv A(20-x) + Bx$$

If $x=0$, $2000 = 20A$

$\boxed{A = 100}$

If $x=20$, $2000 = 20B$

$\boxed{B = 100}$

$$\Rightarrow \int \frac{100}{x} + \frac{100}{20-x}\,dx = \int 1\,dt$$

$$\Rightarrow 100\ln|x| - 100\ln|20-x| = t + C$$

$$\Rightarrow \boxed{100\ln\left|\frac{x}{20-x}\right| + C = t}$$

$t=0$, $x=4 \Rightarrow 100\ln\frac{1}{4} + C = 0$

$C = -100\ln\frac{1}{4}$

$C = 100\ln 4$

$$\Rightarrow 100\ln\left|\frac{x}{20-x}\right| + 100\ln 4 = t$$

$$\Rightarrow 100\left[\ln\left|\frac{x}{20-x}\right| + \ln 4\right] = t$$

$$\Rightarrow t = 100\ln\left|\frac{4x}{20-x}\right|$$

AS REQUIRED

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c)
$$t = 100\ln\left|\frac{4x}{20-x}\right|$$
$$\Rightarrow \frac{1}{100}t = \ln\left|\frac{4x}{20-x}\right|$$
$$\Rightarrow e^{0.01t} = \frac{4x}{20-x}$$
$$\Rightarrow 20e^{0.01t} - xe^{0.01t} = 4x$$
$$\Rightarrow 20e^{0.01t} = xe^{0.01t} + 4x$$
$$\Rightarrow x(e^{0.01t} + 4) = 20e^{0.01t}$$
$$\Rightarrow x = \frac{20e^{0.01t}}{e^{0.01t} + 4}$$
$$\Rightarrow x = \frac{20e^{0.01t}e^{-0.01t}}{e^{0.01t}e^{-0.01t} + 4e^{-0.01t}}$$
$$\Rightarrow x = \frac{20}{1 + 4e^{-0.01t}}$$

AS REQUIRED

d) WHEN $t = 24$

$$x = \frac{20}{1 + 4e^{-0.24}}$$

$$x = 4.82333\ldots$$

$\therefore$ 4823 CHICKENS

$\therefore$ AN <u>EXTRA</u> 823 CHICKENS