C4, IYGB, PAPER J

1. a) $2\cos x + \tan y = 2\sqrt{3}$

Diff w.r.t x

$-2\sin x + \sec^2 y \frac{dy}{dx} = 0$

$\sec^2 y \frac{dy}{dx} = 2\sin x$

$\frac{1}{\cos^2 y}\frac{dy}{dx} = 2\sin x$

$\frac{dy}{dx} = 2\sin x\cos^2 y$

AS REQUIRED

b) $\left.\frac{dy}{dx}\right|_{(\frac{\pi}{6},\frac{\pi}{3})} = 2\sin\frac{\pi}{6}\cos^2\frac{\pi}{3}$

$= 2\times\frac{1}{2}\times\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\therefore$ NORMAL GRADIENT IS -4

$y - y_0 = m(x - x_0)$

$y - \frac{\pi}{3} = -4\left(x - \frac{\pi}{6}\right)$

$y - \frac{\pi}{3} = -4x + \frac{2\pi}{3}$

$y + 4x = \pi$

2. $\int_0^{\frac{\pi}{4}} 4x\cos 4x \, dx$ ... BY PARTS & IGNORING LIMITS

$4x$	4
$\frac{1}{4}\sin 4x$	$\cos 4x$

$\int 4x\cos 4x \, dx = x\sin 4x - \int \sin 4x \, dx$

$\int 4x\cos 4x \, dx = x\sin 4x + \frac{1}{4}\cos 4x + C$

$\therefore \int_0^{\frac{\pi}{4}} 4x\cos 4x \, dx = \left[x\sin 4x + \frac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}}$

$= \left(\cancel{\frac{\pi}{4}\sin\pi} + \frac{1}{4}\cos\pi\right) - \left(0 + \frac{1}{4}\right)$

$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

3. $x = 2t^2 - 1$

$y = 3(t+1)$

$3x - 4y = 3$

SOLVING SIMULTANEOUSLY

$3(2t^2-1) - 4[3(t+1)] = 3$

$6t^2 - 3 - 12t - 12 = 3$

$6t^2 - 12t - 18 = 0$

$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$t = \begin{cases} -1 \\ 3 \end{cases}$

THUS

$$t = \begin{cases} -1 \\ 3 \end{cases} \quad x = \begin{cases} 1 \\ 17 \end{cases} \quad y = \begin{cases} 0 \\ 12 \end{cases}$$

$\therefore (1,0)$ & $(17,12)$

4. a) $$\frac{27x+2}{(2-x)(1+3x)} \equiv \frac{P}{2-x} + \frac{Q}{1+3x}$$

$$27x+2 \equiv P(1+3x) + Q(2-x)$$

IF $x=2$ $\quad 56 = 7P \implies \boxed{P=8}$

IF $x=-\frac{1}{3}$ $\quad -7 = \frac{7}{3}Q \implies \boxed{Q=-3}$

b) $$\frac{27x+2}{(2-x)(1+3x)} = \frac{8}{2-x} - \frac{3}{1+3x} = 8(2-x)^{-1} - 3(1+3x)^{-1}$$

- $8(2-x)^{-1} = 8 \times 2^{-1}(1-\frac{1}{2}x)^{-1} = 4(1-\frac{1}{2}x)^{-1}$

$$= 4\left[1 + \frac{-1}{1}(-\tfrac{1}{2}x)^1 + \frac{-1(-2)}{1\times2}(-\tfrac{1}{2}x)^2 + \frac{-1(-2)(-3)}{1\times2\times3}(-\tfrac{1}{2}x)^3 + O(x^4)\right]$$

$$= 4\left[1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + O(x^4)\right]$$

$$= 4 + 2x + x^2 + \tfrac{1}{2}x^3 + O(x^4)$$

- $-3(1+3x)^{-1} = -3\left[1 + \frac{-1}{1}(3x)^1 + \frac{-1(-2)}{1\times2}(3x)^2 + \frac{-1(-2)(-3)}{1\times2\times3}(3x)^3 + O(x^4)\right]$

$$= -3\left[1 - 3x + 9x^2 - 27x^3 + O(x^4)\right.$$

$$= -3 + 9x - 27x^2 + 81x^3 + O(x^4)$$

THUS

$$\frac{27x+2}{(2-x)(1+3x)} = \begin{array}{l} 4 + 2x + x^2 + \tfrac{1}{2}x^3 + O(x^4) \\ -3 + 9x - 27x^2 + 81x^3 + O(x^4) \\ \hline 1 + 11x - 26x^2 + \tfrac{163}{2}x^3 + O(x^4) \end{array}$$

AS REQUIRED

5. a) $\displaystyle\int \frac{\cos x}{1-\cos x}\,dx = \int \frac{\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx = \int \frac{\cos x(1+\cos x)}{1-\cos^2 x}$

$$= \int \frac{\cos x(1+\cos x)}{\sin^2 x}\,dx = \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin x} \times \frac{1}{\sin x} + \cot^2 x\,dx$$

$$= \int \cot x \operatorname{cosec} x + \cot^2 x\,dx$$

AS REQUIRED

b) USING STANDARD RESULTS $\boxed{\begin{array}{l} \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \end{array}}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1-\cos x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \operatorname{cosec} x + \cot^2 x\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \operatorname{cosec} x + (\operatorname{cosec}^2 x - 1)\,dx = \Big[-\operatorname{cosec} x - \cot x - x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \Big[x + \cot x + \operatorname{cosec} x\Big]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \left[\frac{\pi}{4} + \cot\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{4}\right] - \left[\frac{\pi}{2} + \cancel{\cot\frac{\pi}{2}} + \operatorname{cosec}\frac{\pi}{2}\right]$$

$$= \left(\frac{\pi}{4} + 1 + \sqrt{2}\right) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{4} \cancel{+1} + \sqrt{2} - \frac{\pi}{2} \cancel{-1}$$

$$= \sqrt{2} - \frac{\pi}{4} = \frac{1}{4}\left[4\sqrt{2} - \pi\right]$$

AS REQUIRED

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6. a) WHEN $y=0 \Rightarrow t-t^2=0$

$t(1-t)=0$

$t = \begin{cases} 0 \\ 1 \end{cases} \quad x = \begin{cases} 0 & \leftarrow \text{ORIGIN} \\ 6\times1^2 = 6 \end{cases}$

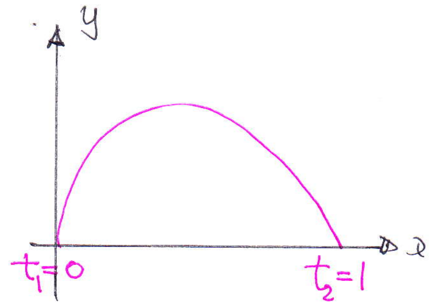
$\therefore P(6,0)$

b) IN CARTESIAN

$$V = \pi\int_{x_1}^{x_2} \left(y(x)\right)^2 dx$$

IN PARAMETRIC IT BECOMES

$$V = \pi\int_{t_1}^{t_2} \left(y(t)\right)^2 \frac{dx}{dt}\, dt$$



$$V = \pi\int_0^1 (t-t^2)^2(12t)\, dt$$

$$V = \pi\int_0^1 12t(t-t^2)^2\, dt$$

AS REQUIRED (I.E $T=1$)

c) $V = \pi\int_0^1 12t(t-t^2)^2 dt = \pi\int_0^1 12t(t^2-2t^3+t^4)\, dt$

$= \pi\int_0^1 12t^3 - 24t^4 + 12t^5\, dt = \pi\left[3t^4 - \frac{24}{5}t^5 + 2t^6\right]_0^1$

$= \pi\left[\left(3 - \frac{24}{5} + 2\right) - 0\right] = \frac{1}{5}\pi$

7. a) $\underline{a} = (0,8,3)$
$\underline{b} = (1,13,1)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (1,13,1) - (0,8,3) = (1,5,-2)$

HENCE $\underline{r} = (0,8,3) + \lambda(1,5,-2)$
$(x,y,z) = (\lambda, 5\lambda+8, 3-2\lambda)$

b) $\underline{r}_2 = (7,0,9) + \mu(2,-3,1)$
$(x,y,z) = (2\mu+7, -3\mu, \mu+9)$

- EQUATE $\underline{i}$ & $\underline{j}$

$\underline{i}: \lambda = 2\mu+7$
$\underline{j}: 5\lambda+8 = -3\mu$

$\Rightarrow 5(2\mu+7)+8 = -3\mu$
$10\mu + 35 + 8 = -3\mu$
$13\mu = -43$
$\boxed{\mu = -\frac{43}{13}}$

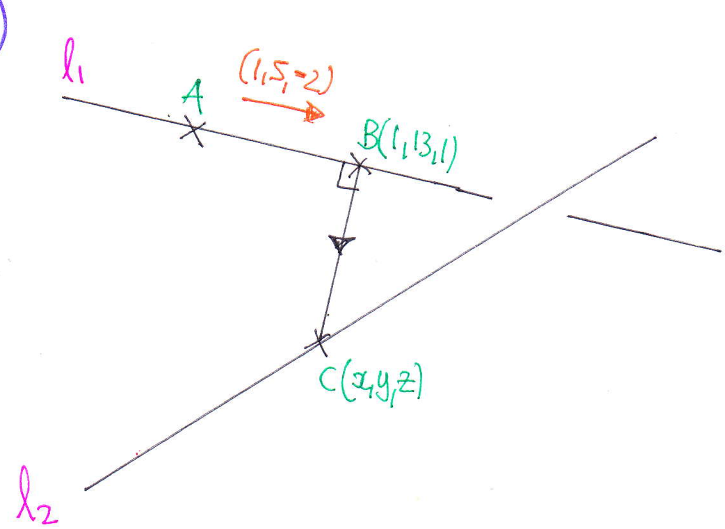
$\lambda = 2\left(-\frac{43}{13}\right)+7 \Rightarrow \boxed{\lambda = \frac{5}{13}}$

CHECK $\underline{k}$

$3-2\lambda = 3-2\times\frac{5}{13} = \frac{29}{13}$
$\mu+9 = -\frac{43}{13}+9 = \frac{74}{13}$

$\frac{29}{13} \neq \frac{74}{13}$ LINES DO NOT INTERSECT

c)



- LET $\underline{c} = (x,y,z)$
$\underline{b} = (1,13,1)$

- $\overrightarrow{BC} = \underline{c} - \underline{b}$
$= (x,y,z) - (1,13,1)$
$= (x-1, y-13, z-1)$

- NOW $\hat{ABC} = 90 \Longrightarrow (x-1, y-13, z-1) \cdot (1,5,-2) = 0$

$$\Longrightarrow x - 1 + 5y - 65 - 2z + 2 = 0$$

$$\Longrightarrow \boxed{x + 5y - 2z = 64}$$

- POINT C LIES ON $l_2 \Longrightarrow (x,y,z) = (2\mu+7, -3\mu, \mu+9)$

$$\boxed{\begin{aligned} x &= 2\mu+7 \\ y &= -3\mu \\ z &= \mu+9 \end{aligned}}$$

THUS $(2\mu+7) + 5(-3\mu) - 2(\mu+9) = 64$

$$\cancel{2\mu} + 7 - 15\mu - \cancel{2\mu} - 18 = 64$$

$$-15\mu = 75$$

$$\boxed{\mu = -5}$$

$\therefore$ $C(-3, 15, 4)$

8. a)

$$\underbrace{\frac{dV}{dt}}_{\text{RATE OF VOLUME}} = \underbrace{+}_{\text{EXPANDS}} \underbrace{k \times \frac{1}{V}}_{\text{INVERSELY PROPORTIONAL}}$$

$$\Longrightarrow \frac{dV}{dt} = \frac{k}{V}$$

$$\Longrightarrow \frac{dV}{dP} \times \frac{dP}{dt} = \frac{k}{V}$$

$$\Longrightarrow \left(-\frac{c}{P^2}\right)\frac{dP}{dt} = \frac{k}{V}$$

$$\Longrightarrow \frac{dP}{dt} = \frac{k}{V} \times \left(-\frac{P^2}{c}\right)$$

$$\Longrightarrow \frac{dP}{dt} = \frac{k}{\frac{c}{P}} \times \frac{-P^2}{c}$$

$$\Longrightarrow \frac{dP}{dt} = \frac{kP}{c} \times \frac{-P^2}{c}$$

$PV = \text{constant}$

$PV = c$

$V = \frac{c}{P}$

$V = cP^{-1}$

$\frac{dV}{dP} = -cP^{-2}$

$\frac{dV}{dP} = -\frac{c}{P^2}$

$$\therefore \frac{dP}{dt} = -AP^3 \quad \left(A = \frac{k}{c^2}\right)$$

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b) SEPARATE VARIABLES

$$-\frac{1}{P^3}\,dP = A\,dt$$

$$\Longrightarrow \int -P^{-3}\,dP = \int A\,dt$$

$$\Longrightarrow \frac{1}{2}P^{-2} = At + C$$

$$\Longrightarrow \boxed{\frac{1}{2P^2} = At + C}$$

When $t=0$, $P=1 \Longrightarrow \frac{1}{2} = C$

$$\Longrightarrow \boxed{\frac{1}{2P^2} = At + \frac{1}{2}}$$

When $t=2$, $P=\frac{1}{3} \Longrightarrow \frac{1}{\frac{2}{9}} = 2A + \frac{1}{2}$

$$\frac{9}{2} = 2A + \frac{1}{2}$$

$$4 = 2A$$

$$A = 2$$

$$\therefore \frac{1}{2P^2} = 2t + \frac{1}{2}$$

$$\frac{1}{P^2} = 4t + 1$$

$$P^2 = \frac{1}{4t+1}$$

AS REQUIRED