FP3 Specimen Paper Herine 2

1. $A = \begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 7-\lambda & 6 \\ 6 & 2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = (7-\lambda)(2-\lambda) - 36$

$= -22 - 9\lambda + \lambda^2 = 0$

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 9\lambda - 22 = 0$

$\therefore (\lambda - 11)(\lambda + 2) = 0$

E. Val:

$\lambda = -2$

$\lambda = 11$

$\Rightarrow$

2. $\frac{9}{2}(e^x + e^{-x}) - \frac{6}{2}(e^x - e^{-x}) = 7$

$\therefore \frac{9}{2}e^x + \frac{9}{2}e^{-x} - 3e^x + 3e^{-x} = 7$

$\therefore \frac{3}{2}e^x + \frac{15}{2}e^{-x} = 7$

$\Rightarrow 3e^x + 15e^{-x} = 14$

$\Rightarrow 3e^{2x} + 15 = 14e^x$

$\Rightarrow 3e^{2x} - 14e^x + 15 = 0$

$(e^x - 3)(3e^x - 5) = 0$ $\quad e^x = \frac{5}{3}$

$\therefore e^x = 3 \Rightarrow x = \ln 3$ $\quad x = \ln\frac{5}{3}$

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3. Let length $= s$

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad s = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(2 - 2\cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{2a^2(1 - \cos t)}\, dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos t = \cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}$

$\cos^2 \frac{t}{2} - (1 - \cos^2 \frac{t}{2})$

$\cos t = 2\cos^2 \frac{t}{2} - 1$

$$= \sqrt{2}\, a \int_0^{2\pi} \sqrt{1 - \cos t}\, dt$$

$$= a\sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2 \frac{t}{2}}\, dt = a\sqrt{2}\sqrt{2} \int \sin \frac{t}{2}\, dt$$

$$= 2a\left[-2\cos \frac{t}{2}\right]_0^{2\pi} = 8a$$

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$= 2a\left(2 - -2\right)$

$= 8a$ $\quad S = 8a$

4. $\int (x^2+4)^{1/2}\,dx$

$u = (x^2+4)^{1/2}$ $\quad u' = \frac{1}{2}(x^2+4)^{-1/2} \times 2x = x\sqrt{x^2+4}$

$v' = 1$ $\quad v = x$

$x\sqrt{(x^2+4)} - \int x^2\sqrt{x^2+4}$

$s^2 + c^2 = 1$

$c^2 - s^2 = 1$

$c^2 = 1 + s^2 = s^2 + 1$

4. $\int (x^2+4)^{1/2}\,dx$

$x = 2\sinh\theta$ $\quad \frac{x}{2} = \sinh\theta$ $\quad \theta = \operatorname{arsinh}\frac{x}{2}$

$x^2 = 4\sinh^2\theta$ $\quad = \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right)$

$\frac{dx}{d\theta} = 2\cosh\theta$

$dx = 2\cosh\theta\,d\theta$

$= \int (4\sinh^2\theta + 4)^{1/2} \cdot 2\cosh\theta\,d\theta$

$= 2\int \sqrt{4(\sinh^2\theta+1)}\cdot\cosh\theta\,d\theta$

$\cos 2x = \cos^2 - \sin^2$

$= 2\int 2\cosh^2\theta\,d\theta$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = 2\cosh^2 x - 1$

$= 4\int \cosh^2\theta\,d\theta$

$\frac{\cosh(2x)+1}{2} = \cosh^2 x$

$= \frac{4}{2}\int \cosh(2\theta) + 1\,d\theta$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= x\sqrt{1+\sinh^2\theta}$

$= x\sqrt{1+\frac{x^2}{4}}$

$= 2\left[\frac{1}{2}\sinh(2\theta) + \theta\right] + C$

$= \sinh 2\theta + 2\theta + C = x\sqrt{1+\frac{x^2}{4}} + 2\ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right) + C$

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5 (a) $y = \arcsin x$

$\therefore \quad \sin y = x$ $\qquad \cos y \equiv \sqrt{1-\sin^2 y} = \sqrt{1-x}$

$\therefore \quad \cos y \frac{dy}{dx} = 1$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ //

$(1-x^2)^{-1/2}$

(b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)$

$= x(1-x^2)^{-3/2}$

$\frac{d^2y}{dx^2} = \frac{x}{(\sqrt{1-x^2})^3}$

$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} = x(1-x^2)^{-3/2}(1-x^2)$

$= x(1-x^2)^{-1/2}$

$= x \cdot \frac{1}{\sqrt{1-x^2}}$

$= x \cdot \frac{dy}{dx}$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

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6.(a).

$$I_n = \int_0^{\pi/2} x^n \sin x \, dx$$

Let $u = x^n \quad u' = nx^{n-1}$

$v' = \sin x \quad v = -\cos x$

$$\therefore \quad I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n \int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$$= 0 + n \int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$u = x^{n-1} \quad u' = (n-1)x^{n-2}$

$v' = \cos x \quad v = \sin x$

$$\therefore \quad I_n = n\left(\left[x^{n-1} \sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2} \sin x\right)$$

$$= n\left(\left(\frac{\pi}{2}\right)^{n-1} - (n-1) I_{n-2}\right)$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

as required.

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(b) $I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6I_1$

$= \frac{3}{4}\pi^2 - 6\int_0^{\pi/2} x \sin x \, dx$

$u = x \quad v' = 1$

$v' = \sin x \quad v = -\cos x$

$= \frac{3}{4}\pi^2 - 6\left[\left[-x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx\right)$

$= \frac{3\pi^2}{4} - 6\left(\left[\sin x\right]_0^{\pi/2}\right)$

$= \frac{3\pi^2}{4} - 6(1-0)$

$= \frac{3\pi^2}{4} - 6$ //

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7(a). $\begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$\det[A(x)] = -2 - x(-2) - 3 = 2x - 5$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$M = \begin{pmatrix} -2 & -2 & 3 \\ 1 & 1 & 1-x \\ 2x & 5 & -3x \end{pmatrix}$

$C = \begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & x-1 \\ 2x & -5 & -3x \end{pmatrix}$

$C^T = \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$

$\therefore [A(x)]^{-1} = \frac{1}{2x-5} \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$

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(b) $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} p+3q-r \\ 3p+2r \\ p+q \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \quad p+3q-r = 2$

$\Rightarrow r = p+3q-2$

$\therefore \quad 3p+2r = 3p+2p+6q-4 = 5p+6q-4$

$5p+6q-4 = 3 \Rightarrow 5p+6q = 7$

$p+q = 4$

$\therefore \quad 5p+5q = 20$

$5p+6q = 7 \Rightarrow 5p+5q+q = 7$

$\Rightarrow 20+q = 7$

$\Rightarrow q = -13$

$\therefore 5p = 20-5(-13) = 85$

$\Rightarrow p = 17$

$r = \frac{3-3p}{2} = -24$

$\therefore \quad p = 17$

$q = -13$

$r = -24$

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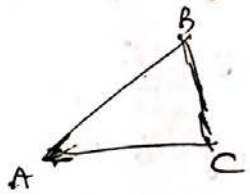
8. (a) $\overrightarrow{AB} = \begin{pmatrix}1\\3\\0\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\-1\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}1\\3\\2\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix}$

$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{matrix}-1 & 3\\ -1 & 3\end{matrix} \times \begin{matrix}-1 & -1\\ 1 & 1\end{matrix} \times \begin{matrix}3 & -1\\ 3 & 1\end{matrix}$

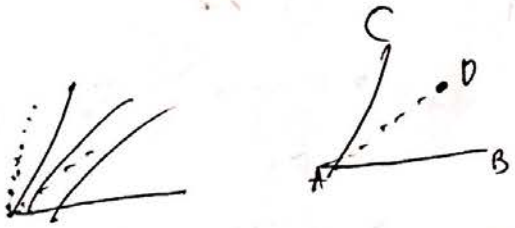
$= \begin{pmatrix}6\\2\\0\end{pmatrix}$



$\therefore$ Area $= \left|\frac{1}{2}\begin{pmatrix}6\\2\\0\end{pmatrix}\right| = \left|\begin{pmatrix}3\\1\\0\end{pmatrix}\right| = \sqrt{3^2+1}$

$= \sqrt{10}$

(b)



$\overrightarrow{AD} = \begin{pmatrix}0\\4\\1\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\4\\0\end{pmatrix}$

Area $= \frac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right|$

$= \frac{1}{6}\left|\begin{pmatrix}-2\\4\\0\end{pmatrix} \cdot \begin{pmatrix}6\\2\\0\end{pmatrix}\right|$

$= \frac{1}{6}\left|-12+8\right| = \frac{2}{3}$

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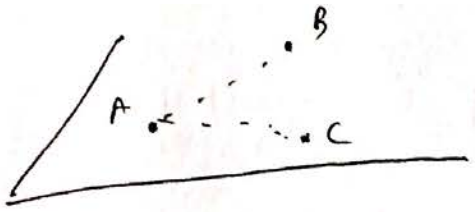
(c)



$\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + k($ ... $D \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = 12$

$\frac{1}{\sqrt{36+4}}$

$\hat{\underset{\sim}{n}} = \frac{1}{\sqrt{40}} \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$

$\Rightarrow$ eqn of plane:

$6x + 2y = 12$

$3x + y = 6$

$3x + y - 6 = 0$

$\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}} = 12$

$D = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$\therefore$ distance $= \left| \frac{3(0) + 4 - 6}{\sqrt{3^2 + 1^2}} \right|$

$= \left| \frac{-2}{\sqrt{10}} \right| = \frac{\sqrt{10}}{5}$ //

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9(a). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{2}{a^2}x - \frac{2}{b^2}y\frac{dy}{dx} = 0$

$\therefore \frac{2}{a^2}x = \frac{2}{b^2}y\frac{dy}{dx}$

$\frac{1}{a^2}x = \frac{1}{b^2}y\frac{dy}{dx}$

$\frac{b^2}{a^2}\frac{x}{y} = \frac{dy}{dx}$

$\frac{1}{\frac{\sin}{\cos}}$

$\left(\frac{dy}{dx}\right)_{\substack{x = a\sec\theta \\ y = b\tan\theta}} = \frac{b^2}{a^2} \cdot \frac{a\sec\theta}{b\tan\theta} = \frac{b\cancel{a}}{\cancel{a}\cancel{b}}\operatorname{cosec}\theta$

$= \frac{b}{a}\operatorname{cosec}\theta$

$\therefore m_N = \frac{-1}{\frac{b}{a}\operatorname{cosec}\theta} = -\frac{1}{\frac{b}{a}}\sin\theta = -\frac{a}{b}\sin\theta$

$\therefore y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$

$\therefore by\tan\sin\theta = a^2\frac{\sin\theta}{\cos\theta} + b^2\tan\theta$

$by\tan\sin\theta = a^2\tan\theta + b^2\tan\theta$ as required

$\therefore$ Equation of $= (a^2 + b^2)\tan\theta$

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(b)

when $x=0$

$by = (a^2+b^2)\tan\theta$

$\Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$

$\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

when $y=0$

$a\sin\theta = (a^2+b^2)\tan\theta$

$\frac{\sin}{\cos}$

$\therefore x = \frac{(a^2+b^2)}{a}\sec\theta$

$\therefore \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$

$\therefore M = \left(\frac{a^2+b^2}{2a}\sec\theta, \frac{a^2+b^2}{2b}\tan\theta\right)$

$x = \frac{a^2+b^2}{2a}\sec\theta \qquad y = \frac{a^2+b^2}{2b}\tan\theta$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{c^2}$

$t^2+1=\sec^2$

$\sec\theta = \sqrt{\tan^2\theta+1} \quad \& \quad \frac{2by}{a^2+b^2} = \tan\theta$

$\therefore x = \frac{a^2+b^2}{2a}\sqrt{\tan^2\theta+1} = \frac{a^2+b^2}{2a}\sqrt{\frac{4b^2y^2}{(a^2+b^2)^2}+1}$

$\therefore x^2 = \frac{(a^2+b^2)^2}{4a^2}\left(\frac{4b^2y^2}{(a^2+b^2)^2}+1\right)$

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$$x^2 = \frac{4b^2y^2}{4a^2} + \frac{(a^2+b^2)^2}{4a^2}$$

$$x = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2} \quad \times 4a^2$$

$$x^2 = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2}$$

$$4a^2x^2 = 4b^2y^2 + (a^2+b^2)^2$$

$$\therefore 4a^2x^2 - 4b^2y^2 = (a^2+b^2)^2$$

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