Cambridge International A Level

Cambridge Assessment International Education
Cambridge International Advanced Level

FURTHER MATHEMATICS	**9231/21**
Paper 2	**October/November 2019**
MARK SCHEME	
Maximum Mark: 100	

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2019 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more “method” steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$(T-1)^4 / 2 = 8$	M1 A1	Equate radial acceln. to 8 at $t = T$ from v^2/r
	$T = 3$ (or $T - 1 = 2$)	A1	Hence find positive value of T (or of $T - 1$)
	$a_T = 2\,(T-1) = 4$ [m s^{-2}]	M1 A1	Find magnitude of transverse acceleration at $t = T$
		5	

Question	Answer	Marks	Guidance
2(i)	$R_E \times 3a = W \cos \theta \times 2a - W \sin \theta \times 2a$ or $R_E \times 3a = W \times 2a\,(1 - \tan \theta) \cos \theta$ or $R_E \times 3a = 2\sqrt{2}W \sin\left(\frac{\pi}{4} - \theta\right)$	M1 A1	Take moments about B
	$R_E = 4W / 3\sqrt{10}$	A1	Find normal reaction at E. AEF
	$R_B = W - R_E \cos \theta = 3W/5$	M1 A1	Find normal reaction at B by resolving forces vertically
		5	
2(ii)	$F_B = R_E \sin \theta = 2W/15$	M1 A1	Find friction at B by resolving forces horizontally
	$\mu = (2/15) / (3/5) = 2/9$	A1	Find μ from $F_B = \mu R_B$
		3	

Question	Answer	Marks	Guidance
3(i)	$5mv_A + 5mv_B = 5mu$ [$v_A + v_B = u$] and $v_B - v_A = e\,u$	M1	Use consvn. of momentum for *A* and *B* and use Newton's restitution law with consistent LHS signs. AEF
	$v_A = ½\,(1 - e)\,u$	A1	Combine to verify speed of *A*. AG
	$v_B = ½\,(1 + e)\,u$	A1	Find speed of *B*
		3	
3(ii)	$5mv_B' + 3mv_C = 5mv_B$ [$5v_B' + 3v_C = 5v_B$] $v_C - v_B' = ev_B$	M1	Use consvn. of momentum for *B* and *C* and use Newton's restitution law with consistent LHS signs. AEF
	$v_B' = (1/8)\,(5 - 3e)\,v_B$ [$v_C = (1/8)\,(5 + 5e)\,v_B$]	A1	Combine to find v_B' (v_C not reqd as *B*, *C* cannot collide again)
	$\frac{1}{2}(1 - e)\,u \leqslant (1/8)\,(5 - 3e) \times \frac{1}{2}\,(1 + e)\,u$	M1	Find condition on *e* using $v_A \leqslant v_B'$
	$3e^2 - 10e + 3 \leqslant 0$	A1	Simplify to a quadratic inequality
	$\frac{1}{3} \leqslant e$	A1	Solve to give a lower bound on *e*
	$\frac{1}{3} \leqslant e \leqslant 1$	A1	Non-strict inequality
		6	

Question	Answer	Marks	Guidance
4(i)	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga\cos\theta$	M1	Use conservation of energy to slack point P_1
	$mv^2/a - mg\cos\theta = 0$	M1 A1	Equate tension at P_1 to 0 by using $F = ma$ A1 if both eqns correct, with m included. AG
	$v^2 = 2ag - 2ag\cos\theta = ag\cos\theta$	M1	Combine to verify $\cos\theta$ using $u = \sqrt{(2ag)}$
	$\cos\theta = 2/3$	A1	
		5	
4(ii)	$v_V = v\sin\theta = \sqrt{(2ag/3)}\,(\sqrt{5}/3)$ or $v_V^2 = (10/27)\,ag$	M1	Find vertical speed v_V at P_1
	$h = v_V^2/2g = (5/27)\,a$ or $0{\cdot}185\,a$	M1 A1	Find height risen above P_1 by considering vertical motion
	$h + a\cos\theta = (23/27)\,a$ or $0{\cdot}852\,a$	A1	Find total height risen above level of O
		4	

Question	Answer	Marks	Guidance
5(i)	$I_{rod} = \frac{1}{3}\ \lambda Ma^2 + \lambda M(a/2)^2$ [= (7/12) λMa^2]	B1	Find or state MI of rod *AB* about axis *L*
	$I_O = \frac{2}{3}\ 3M\ a^2 + 3M\ (5a/2)^2$ [= (83/4) Ma^2]	M1 A1	Find MI of hollow sphere centre *O* about axis *L*
	$I_C = (2/5)\ 5Ma^2 + 5M\ (3a/2)^2$ [= (53/4) Ma^2]	M1 A1	Find MI of solid sphere centre *C* about axis *L*
	$I = (7\lambda/12 + 83/4 + 53/4)\ Ma^2$ $= ((7\lambda + 408) / 12)\ Ma^2$	A1	Verify MI of object about axis *L*. AG
		6	
5(ii)	[–] $I\ \mathrm{d}^2\theta/\mathrm{d}t^2 = [-\ 3Mg \times (5a/2) \sin\theta + 5Mg \times (3a/2) \sin\theta]$ $-\ \lambda Mg \times (a/2) \sin\theta$	M1 A1	Use eqn of circular motion to find $\mathrm{d}^2\theta/\mathrm{d}t^2$ where θ is angle of rod with vertical. AEF
	$\mathrm{d}^2\theta/\mathrm{d}t^2 = -\ \{6g\lambda / (7\lambda + 408)a\}\ \theta$	M1*	Approximate $\sin\theta$ by θ to give standard form of SHM eqn
	$T = 2\pi\ \sqrt{\{(7\lambda + 408)a / 6g\lambda\}} = 5\pi\sqrt{(2a/g)}$	DM1A1	Find possible values of λ by equating period *T* to $5\pi\sqrt{(2a/g)}$.AEF
	$\lambda = 6$	A1	
		6	

Question	Answer	Marks	Guidance
6(i)	$t\sqrt{(s^2/9)} = ½\,(1{\cdot}85 - 1{\cdot}65)$ [$= 0{\cdot}1$]	M1	Find estimate s^2 of population variance (must be t)
	$t_{8,\,0.975} = 2{\cdot}306$ (to 3 s.f.)	A1	Use of correct tabular t-value
	$s^2 = 9 \times 0{\cdot}04337^2 = 0{\cdot}0169$ *or* $0{\cdot}130[1]^2$	A1	
		3	
6(ii)	$\bar{x} = \frac{1}{2}\,(1{\cdot}65 + 1{\cdot}85) = 1{\cdot}75$ or $\Sigma x = 9 \times 1{\cdot}75 = 15{\cdot}75$	M1 A1	Find sample mean $\bar{x}$
	$s^2 = (\Sigma x^2 - 9 \times \bar{x}^2) / 8$ or $\{\Sigma x^2 - (\Sigma x)^2 / 9\} / 8$	M1	*or* Σx
	$\Sigma x^2 = 8 \times 0{\cdot}0169 + 15{\cdot}75^2/9 = 27{\cdot}7$	A1	Find Σx^2 from s^2
		4	

Question	Answer	Marks	Guidance
7(i)	$a = 1/200$ *or* $0{\cdot}005$	B1	State a or find a by equating mean value to $1/a$
		1	
7(ii)	$p = P(T < 150) = F(150) = 1 - e^{-150a}$	M1	Find $P(T < 150)$
	$p = 1 - e^{-0{\cdot}75} = 0{\cdot}528$	A1	
		2	
7(iii)	$1 - p^n > 0{\cdot}99$	M1	Formulate condition for n
	$0{\cdot}01 > (1 - e^{-0{\cdot}75})^n$ or $0{\cdot}01 > 0{\cdot}528^n$	A1	
	$n > \log 0{\cdot}01 / \log 0{\cdot}528$	M1	Rearrange and take logs to give bound
	$n > 7{\cdot}20$ [or $7{\cdot}21$] so $n_{min} = 8$	A1	Find n_{min}
		4	

Question	Answer	Marks	Guidance
8	$\bar{x}_A = 32{\cdot}4 / 8 = 4{\cdot}05$	B1	Find sample mean for *A*
	$s_A^2 = (131{\cdot}82 - 32{\cdot}4^2/8) / 7$ $s_A^2 = 3/35$ (or $0{\cdot}08571$ or $0{\cdot}2928^2$ both to 3 s.f.)	M1	Estimate or imply popln. variance for *A*
	H_0: $\mu_A = \mu_B$, H_1: $\mu_A \neq \mu_B$	B1	State hypotheses. AEF
	$s^2 = (7\, s_A^2 + 9\, s_B^2) / 16 = 0{\cdot}12497$ or $0{\cdot}3535^2$ or $\frac{3999}{32000}$	M1 A1	Estimate (pooled) common variance
	$t_{16,\, 0.95} = 1{\cdot}746$	B1*	State or use correct tabular *t* value
	[–] $t = (x_A - \bar{x}_B) / s\, \sqrt{(1/8 + 1/10)}$	M1	
	$= 0{\cdot}27 / 0{\cdot}1677 = 1{\cdot}61$	A1	Find value of *t* (or can compare $\bar{x}_A - \bar{x}_B = 0{\cdot}27$ with $0{\cdot}293$)
	$t < 1{\cdot}75$ so [accept H_0] mean masses are the same	DB1	Correct conclusion (FT on *t*, dep B1*). AEF
		9	

Question	Answer	Marks	Guidance
9(i)	$\Sigma x = 15$, $\Sigma y = 7 + p + q$, $\Sigma xy = 17 + 2p + 3q$ $\Sigma x^2 = 55$, $[\Sigma y^2 = 21 + p^2 + q^2]$	M1	Find required summations
	$S_{xx} = 55 - 15^2 / 5 = 10$ and $S_{xy} = 17 + 2p + 3q - 15 \times (7 + p + q) / 5 = -4 - p$	M1 A1	
	$-0{\cdot}5 = S_{xy} / S_{xx} = (-4 - p) / 10$ $\quad p = 1$	M1 A1	Find p from gradient in eqn. of regression line
	$(7 + p + q) / 5 = -0{\cdot}5 \times 15/5 + 3{\cdot}5$ $\quad q = 2$	M1 A1	Find q from means and regression line
		7	
9(ii)	$\Sigma y = 10$, $\Sigma y^2 = 26$, $S_{yy} = 26 - 10^2/5 = 6$	M1	Find S_{yy}
	$r = S_{xy} / \sqrt{(S_{xx} S_{yy})} = -5 / \sqrt{(10 \times 6)}$	M1	Find correlation coefficient r
	$r = -0{\cdot}645[5]$ [allow $-0{\cdot}646$]	A1	
		3	

Question	Answer	Marks	Guidance
10(i)	$F(x) = \int f(x)\,dx = (1/30)(-8/x + x^3 - 14x)$ [$+ c$]	**M1**	Find or state distribution function $F(x)$ for $2 \leqslant x \leqslant 4$
	$F(x) = (1/30)(-8/x + x^3 - 14x + 24)$	**M1**	Using $F(2) = 0$ or $F(4) = 1$ to find c if necessary. AEF
	$F(x) = 0$ ($x <$ *or* $\leqslant 2$), $F(x) = 1$ ($x >$ *or* $\geqslant 4$)	**A1**	State $F(x)$ for other values of x
		3	
10(ii)	$G(y) = P(Y < y) = P(X^2 < y)$ $G(y) = P(X < \sqrt{y}) = F(\sqrt{y})$ $G(y) = (1/30)\left(-8/y^{\frac{1}{2}} + y^{\frac{3}{2}} - 14y^{\frac{1}{2}} + 24\right)$	**M1** **A1**	Find or state $G(y)$ for $2 \leqslant x \leqslant 4$ from $Y = X^2$ (allow $<$ or $\leqslant$ throughout)
	Alternative method for question 10(ii)		
	Use $x = y^{\frac{1}{2}}$ to find $f(x) = (1/30)(8/y + 3y - 14), \dfrac{dx}{dy} = -\dfrac{1}{2}y^{-\frac{1}{2}}$	**(M1** **A1)**	Find $f(x)$ *and* $\dfrac{dx}{dy}$ for use in $g(y) = f(x) \times \left\|\dfrac{dy}{dx}\right\|$
	$g(y)[= G'(y)] = (1/30)\left(4/y^{\frac{3}{2}} + (3/2)y^{\frac{1}{2}} - 7/y^{\frac{1}{2}}\right)$ for $4 \leqslant y \leqslant 16$ [$g(y) = 0$ otherwise]	**A1** **A1**	Find $g(y)$. AEF State corresponding range of y for $G(y)$ *or* $g(y)$
		4	
10(iii)	$(1/30)\left(-8/y^{\frac{1}{2}} + y^{\frac{3}{2}} - 14y^{\frac{1}{2}} + 24\right) = 0.8$	**M1**	Set $G(y) = 0{\cdot}8$
	$-8 + y^2 - 14y = 0$, $y = 7 + \sqrt{57}$ *or* $14{\cdot}5[5]$ [rejecting $7 - \sqrt{57}$; allow $14{\cdot}6$]	**M1 A1**	Rearrange to give quadratic in y and solve to find value of y
		3	

Question	Answer	Marks	Guidance
11A(i)	$10(AP - 0{\cdot}6) / 0{\cdot}6 = 20(1{\cdot}2 - AP - 0{\cdot}4) / 0{\cdot}4$	**M1 A1**	Verify AP by equating equilibrium tensions. AEF
	$4AP - 2{\cdot}4 = 9{\cdot}6 - 12AP$ $AP = 0{\cdot}75$ [m]	**A1**	AG
		3	
11A(ii)	$m\,d^2x/dt^2 = -10(0{\cdot}15 + x) / 0{\cdot}6 + 20(0{\cdot}05 - x) / 0{\cdot}4$ or $m\,d^2x/dt^2 = +10(0{\cdot}15 - x) / 0{\cdot}6 - 20(0{\cdot}05 + x) / 0{\cdot}4$	**M1** **A1 A1**	Apply Newton's law at $0{\cdot}75 + x$ or $0{\cdot}75 - x$ from A (M1 requires LHS and 2 tensions: A1 for each correct tension)
	$\frac{2}{3}\,d^2x/dt^2 = -(80 / 1{\cdot}2)\,x$, $d^2x/dt^2 = -100\,x$	**M1 A1**	Simplify to give SHM eqn. in standard form
	$T = 2\pi/\omega = 2\pi/10 = \pi/5$ *or* $0{\cdot}628$ [s]	**DB1**	State the period T with FT on ω from SHM eqn.
		6	
11A(iii)	$(a = 0{\cdot}75 - 0{\cdot}7 = 0{\cdot}05)$ $v_{max} = \omega \times a$	**M1**	Find speed at equilibrium position from ωa
	$v_{max} = 0{\cdot}5$ [m s^{-1}]	**A1**	
		2	
11A(iv)	$x = a/2 = 0{\cdot}025$	**M1**	Find value of x giving half max. acceln.
	$v = \omega\sqrt{(a^2 - x^2)} = 10\sqrt{(0{\cdot}05^2 - 0{\cdot}025^2)}$	**M1**	
	$v = 0{\cdot}433$ [m s^{-1}]	**A1**	Find corresponding speed
		3	

Question	Answer	Marks	Guidance
11B(i)	$\bar{x} = (1/40)\ \Sigma\ x\ \mathrm{f}(x) = 68/40 = 1{\cdot}7$	**B1**	Find mean of sample
	$(1/40)\ \Sigma\ x^2\ \mathrm{f}(x) = 178/40 = 4{\cdot}45$, Var $= 4{\cdot}45 - 1{\cdot}7^2 = 1{\cdot}56$	**B1**	Find variance of sample
	Mean and variance are similar so Poisson may be suitable	**B1**	State valid comment
		3	
11B(ii)	$a = 40 \times 1{\cdot}6^5\ \mathrm{e}^{-1{\cdot}6}/5! = 40 \times 0{\cdot}01764$	**M1**	AG
	$a = 0{\cdot}706$	**A1**	Verify a from Poisson term
	$b = 40 - 39{\cdot}758 = 0{\cdot}242$	**B1***	Find b
		3	
11B(iii)	H_0: Distribution fits/models data	**B1**	State (at least) null hypothesis in full
	O_i: 6 15 9 $\underline{10}$ E_i: 8·076 12·921 10·337 $\underline{8{\cdot}666}$	**DM1** **A1**	Combine values consistent with all exp. values ⩾ 5 (FT on b, dep B1*)
	$X^2 =$ 0·5337 + 0·3345 + 0·1729 + 0·2053 = 1·25	**M1**	Find value of X^2 from $\Sigma\ (E_i - O_i)^2 / E_i$ [or $\Sigma\ O_i^2/E_i - n$]
	$X^2 = 1{\cdot}25$	**A1**	
	No. n of cells: 7 6 5 $\underline{4}$ 3 $\chi_{n-1,\,0.9}{}^2$: 10·64 9·236 7.779 $\underline{6{\cdot}251}$ 4·605	**DB1**	State or use consistent tabular value $\chi_{n-1,\,0.9}{}^2$ (to 3 s.f.) [FT on number, n, of cells used to find X^2]
	Accept H_0 if X^2 < tabular value (using their values)	**M1**	AEF
	1·25 [± 0·1] < 6·25 so distn. fits [data] or distn. is a suitable model	**A1**	Conclusion (requires both values approx. correct). AEF
		8	