

**CAMBRIDGE INTERNATIONAL EXAMINATIONS**

Cambridge International Advanced Level

## **MARK SCHEME for the October/November 2015 series**

### **9231 FURTHER MATHEMATICS**

**9231/23**

Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2015 series for most Cambridge IGCSE<sup>®</sup>, Cambridge International A and AS Level components and some Cambridge O Level components.

® IGCSE is the registered trademark of Cambridge International Examinations.

Question Number	Mark Scheme Details	Part Mark	Total Mark
1	<p>Find 3 independent equations for <math>T, R_A, R_B</math>:</p> <p>Resolve horizontally: <math>R_B = T \cos \alpha</math> <b>M1 A1</b></p> <p>Resolve vertically: <math>R_A = W + T \sin \alpha</math> <b>M1 A1</b></p> <p>Take moments about <math>A</math>: (<math>a</math> may be omitted from moment eqns)  <math>R_B 3a \sin \theta = W (3a/2) \cos \theta</math>  <math>+ T a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)</math>  <i>or</i> <math>+ T a \sin (\alpha + \theta)</math>  <i>or</i> <math>+ T 3a \cos \theta \sin \alpha</math> <b>M1 A1</b></p> <p>Take moments about <math>B</math>:  <math>R_A 3a \cos \theta = W (3a/2) \cos \theta</math>  <math>+ T 2a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)</math>  <i>or</i> <math>+ T 2a \sin (\alpha + \theta)</math>  <i>or</i> <math>+ T 3a \sin \theta \cos \alpha</math> <b>(M1 A1)</b></p> <p>Take moments about <math>C</math>:  <math>R_A a \cos \theta + W (a/2) \cos \theta</math>  <math>= R_B 2a \sin \theta</math> <b>(M1 A1)</b></p> <p>Take moments about <math>D</math>:  <math>R_A 3a \cos \theta - W (3a/2) \cos \theta</math>  <math>= R_B 3a \sin \theta</math> <b>(M1 A1)</b></p> <p>Solve for <math>T, R_A, R_B</math> (AEF in <math>W</math> and <math>\alpha</math>):  <math>T = W / 2 \sin \alpha</math> <i>or</i> <math>\frac{1}{2}W \operatorname{cosec} \alpha</math> <b>B1</b>  <math>R_A = 3W / 2</math> <b>B1</b>  <math>R_B = W / 2 \tan \alpha</math> <i>or</i> <math>\frac{1}{2}W \cot \alpha</math> <b>B1</b></p>	9	9
2	<p>For <math>A</math> &amp; <math>B</math> use conservation of momentum, e.g.: <math>2mv_A + mv_B = 2mu</math> (allow <math>2v_A + v_B = 2u</math>) <b>M1</b></p> <p>Use Newton's law of restitution (consistent signs): <math>v_B - v_A = eu</math> <b>M1</b></p> <p>Combine to find <math>v_A</math> and <math>v_B</math>:  <math>v_A = (2 - e)u/3, v_B = 2(1 + e)u/3</math> <b>A1, A1</b></p> <p>Find <math>e</math> from <math>v_A =  v_B' </math> with <math>v_B' = [-] 0.4 v_B</math>:  <math>(2 - e) = 0.8(1 + e), e = 2/3</math> <b>M1 A1</b></p> <p><i>EITHER</i>: Equate times in terms of reqd. distance <math>x</math>: <math>(d - x)/v_A = d/v_B + x/v_B'</math> (AEF) <b>M1 A1</b>  [speeds need not be found:  <math>v_A = v_B' = 4u/9, v_B = 10u/9</math>  Use <math>v_A = v_B' = 0.4 v_B</math> to solve for <math>x</math>:  <math>d - x = 0.4 d + x, x = 0.3 d</math> <b>M1 A1</b></p> <p><i>OR</i>: Find dist. moved by <math>A</math> when <math>B</math> reaches wall:  <math>d_A = (d/v_B) v_A = 0.4 d</math> <b>(M1 A1)</b></p> <p>Find reqd. distance <math>x</math>:  <math>x = \frac{1}{2}(d - d_A) = 0.3 d</math> <b>(M1 A1)</b></p>	4	10

Question Number	Mark Scheme Details	Part Mark	Total	
3	Find $k$ by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg(a/2)/a = 2mg(3a/2 - ka)/ka$ $1/2 = 3/k - 2, \quad k = 6/5 \text{ or } 1.2$	M1 A1 A1	3
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m \frac{d^2x}{dt^2} = -mg(a/2 + x)/a$ $+ 2mg(3a/2 - ka - x)/ka$ or $m \frac{d^2y}{dt^2} = +mg(a/2 - y)/a$ $- 2mg(3a/2 - ka + y)/ka$	M1 A2	
	Simplify to give standard SHM eqn, e.g.: S.R.: B1 if no derivation (max 2/5)	$\frac{d^2x}{dt^2} = - (1 + 2/k)gx / a$ $= - 8gx/3a$	A1	
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$ : $T = 2\pi\sqrt{(3a/8g)}$ or $\pi\sqrt{(3a/2g)}$ ( $\sqrt$ on $\omega$ )	or $3.85\sqrt{(a/g)}$ or $1.22\sqrt{a}$ [s]	B1 <sup>4</sup>	5
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$ :	$0.7^2 = (8g/3a)\{(0.2a)^2 - (0.05a)^2\}$	M1 A1	
	Solve to find numerical value of $a$ :	$0.49 = (8g/3) \times 0.0375a, \quad a = 0.49$	A1	3
			<b>11</b>	

Question Number	Mark Scheme Details	Part Mark	
4	<i>EITHER:</i> Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$ <b>B1</b>		
	<i>OR:</i> Use energy at e.g. $\theta$ to upward vertical: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta)$ Find tension $T$ by using $F = ma$ radially: $T = mv^2/a - mg \cos \theta$ Eliminate $v^2$ : $= mu^2/a + mg(2 - 3 \cos \theta)$ Find $T$ at top by taking $\theta = 0$ : $T = mu^2/a - mg$ <b>(B1)</b>		
	Find $u_{\min}$ by requiring $T \geq 0$ at top [ $T > 0$ ]: $u^2/a - g \geq 0$ so $u_{\min} = \sqrt{ag}$ <b>A.G.</b> <b>B1</b>		2
	Find $v$ at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ $v^2 = ag + 4ag, v = \sqrt{5ag}$ <b>M1</b> <b>A1</b>		
	Find new speed $V$ from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ <b>M1</b> $V = 4v/5 = 4\sqrt{ag/5}$ or $(4/5)\sqrt{5ag}$ <b>A1</b> <b>AEF</b>		4
	Find $w^2$ at angle $\theta$ from conservation of energy: $\frac{1}{2}m'w^2 = \frac{1}{2}m'V^2 - m'ga(1 + \cos \theta)$ (condone $m$ instead of $m'$ here since cancels out) $[w^2 = ag(6/5 - 2 \cos \theta)]$ <b>M1</b> <b>A1</b>		
	<b>S.R.</b> Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3 \cos \theta)$ ] $\frac{1}{2}m'w^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta) - \frac{1}{4}mga(1 + \cos \theta)$ <b>(B1)</b>		
	Find tension $T'$ there by using $F = ma$ radially: $T' = m'w^2/a - m'g \cos \theta$ <b>B1</b>		
	Eliminate $w^2$ : $= m'V^2/a - m'g(2 + 3 \cos \theta)$ <b>A1</b>		
	Substitute for $m'$ and $V$ : $= (5mg/4)(6/5 - 3 \cos \theta)$ <b>AEF</b> or $3mg/2 - (15/4)mg \cos \theta$ <b>A1</b>		5
Find $\cos \theta$ when string becomes slack from $T' = 0$ : $\cos \theta = \frac{1}{3} \times 6/5 = 2/5$ or 0.4 <b>M1</b> <b>A1</b> <b>S.R.</b> Allow if found from $T' = mg(6/5 - 3 \cos \theta)$		2	13
5	Find or use sample mean <u>and</u> estimate population variance: (allow biased here: 0.412 or 0.642 <sup>2</sup> ) $\bar{x} = 222.8 / 10 = 22.28$ $s^2 = 4.12 / 9 = 0.458$ or $103/225$ or $0.677^2$ <b>M1</b>		
	Find confidence interval (allow $z$ in place of $t$ ) e.g.: $22.28 \pm t\sqrt{(0.458 / 10)}$ <b>M1</b> <b>A1</b>		
	Use of correct tabular value: $t_{9, 0.975} = 2.26[2]$ <b>A1</b>		
	Evaluate C.I. correct to 3 s.f.: $22.3 \pm 0.48[4]$ or $[21.8, 22.8]$ <b>A1</b>		5

Page 5	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2015	9231	23

Question Number	Mark Scheme Details	Part Mark	Total Mark
<b>6</b>	Find prob. $p$ of head from mean = $2 \times$ variance: $1/p = 2 \times (1-p)/p^2$ , $p = 2/3$ <b>A.G.</b> <b>M1 A1</b>	2	<b>8</b>
	<b>(i)</b> Find $P(X = 4)$ (denoting $1 - p$ by $q$ [= $1/3$ ]): $P(X = 4) = q^3 \times p$ $= 2/81$ or 0.0247 <b>B1</b>	1	
	<b>(ii)</b> Find or state $P(X > 4)$ : $P(X > 4) [= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)] = q^4$ $= 1/81$ or 0.0123 <b>M1 A1</b>	2	
	<b>(iii)</b> Formulate condition for $N$ : Take logs (any base) to give bound for $N$ : Find $N_{\min}$ : ( $N < 6.29$ or $N = 6.29$ earns M2 A0) $1 - q^N > 0.999$ , $[(1/3)^N < 0.001]$ <b>M1</b> $N > \log 0.001 / \log 1/3$ <b>M1</b> $N > 6.29$ , $N_{\min} = 7$ <b>A1</b>	3	
<b>7</b>	Find $F(x)$ for $1 \leq x \leq 4$ : $F(x) = (x^3 - 1)/63$ <b>B1</b>	5	<b>9</b>
	Find $G(y)$ from $Y = X^2$ for $1 \leq x \leq 4$ : (result may be stated) $G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$ <b>M1 A1</b>		
	Find $g(y)$ for corresponding range of $y$ : $g(y) = y^{1/2}/42$ <b>A.G.</b> <b>A1</b>		
	Find or state corresponding range of $y$ : $1 \leq y \leq 16$ <b>A.G.</b> <b>B1</b>		
	<b>(i)</b> Find median value $m$ of $Y$ : $(m^{3/2} - 1)/63 = 1/2$ $m = 32.5^{2/3} = 10.2$ <b>M1 A1</b>		
<b>(ii)</b> Find $E(Y)$ [or equivalently $E(X^2)$ ]: $E(Y) = \int y g(y) dy = \int y^{3/2} dy / 42$ $= [y^{5/2}]_1^{16} / 105 = 1023/105$ $= 341/35$ or 9.74 <b>M1 A1</b>	2		

Question Number	Mark Scheme Details	Part Mark	Total Mark
8	<p>Find mean of sample data [for use in Poisson distn.]:  <math>\lambda = 220/100 = 2.2</math> <b>B1</b></p> <p>State (at least) null hypothesis (AEF):  <math>H_0</math>: Poisson distn. fits data  or <math>\lambda = 2.2</math> <b>B1</b></p> <p>Find expected values <math>100\lambda^r e^{-\lambda}/r!</math> (to 1 d.p.):  (ignore incorrect final value here for M1)  11.080 24.377 26.814 19.664  10.8151 4.759 2.491 <b>M1 A1</b></p> <p>Combine last two cells so that exp. value <math>\geq 5</math>:  <math>O_i</math>: 3  <math>E_i</math>: 7.25 <b>M1*</b></p> <p>Calculate value of <math>\chi^2</math> (to 2 d.p.; A1 dep M1*):  (allow 7.95 if 1 d.p. exp.values used)  <math>\chi^2 = 0.076 + 2.879 + 0.653 + 1.448</math>  <math>+ 0.441 + 2.491</math>  <math>= 7.99</math> <b>M1 A1</b></p> <p>State or use consistent tabular value (to 3 s.f.):  5 cells: <math>\chi_{3,0.95}^2 = 7.815</math>  6 cells: <math>\chi_{4,0.95}^2 = 9.488</math> (correct) <b>B1</b>  7 cells: <math>\chi_{5,0.95}^2 = 11.07</math></p> <p>State or imply valid method for conclusion e.g.:  Accept <math>H_0</math> if <math>\chi^2 &lt;</math> tabular value <b>M1</b></p> <p>Conclusion (AEF, requires both values correct):  Distn fits or <math>\lambda = 2.2</math> <b>A1</b>  Not combining cells [so <math>\chi^2 = 8.64</math>] can earn  B1 B1 M1 A1 M0 M1 B1 M1 (max 7)</p>	10	10
9	<p>Calculate gradient <math>b_1</math> in <math>y - \bar{y} = b_1(x - \bar{x})</math>:  <math>S_{xy} = 24\ 879 - 472 \times 400/8</math>  <math>= 1\ 279</math>  <math>S_{xx} = 29\ 950 - 472^2/8 = 2\ 102</math>  <math>b_1 = S_{xy} / S_{xx} = 0.6085</math> (3 s.f.) <b>M1 A1</b></p> <p>Find regression line of y on x:  <math>y = 400/8 + b_1(x - 472/8)</math> <b>M1 A1</b>  <math>= 50 + 0.6085(x - 59)</math>  <math>= 0.6085x + 14.1</math></p> <p>Find y when <math>x = 72</math>:  <math>= 57.9</math> or 58</p> <p>Allow use of regression line of x on y  (since neither variable clearly independent):  <math>S_{yy} = 21\ 226 - 400^2/8 = 1\ 226</math>  <math>b_2 = S_{xy} / S_{yy} = 1.043</math> <b>(M1 A1)</b>  <math>x = 472/8 + b_2(y - 400/8)</math> <b>(M1 A1)</b>  <math>= 1.043y + 6.85</math>  <math>Y = 62.5</math> or 62 <b>(A1)</b></p>	5	

Page 7	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2015	9231	23

Question Number	Mark Scheme Details	Part Mark	Total
	<p>Find product moment correlation coefficient <math>r</math>: <math>r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}</math>  <i>or</i> <math>\sqrt{(0.6085 \times 1.043)} = 0.797</math> <b>M1 A1*</b></p> <p>State both hypotheses (B0 for <math>r \dots</math>): <math>H_0: \rho = 0, H_1: \rho \neq 0</math> <b>B1</b></p> <p>State or use correct tabular two-tail <math>r</math>-value: <math>r_{8,5\%} = 0.707</math> <b>B1*</b></p> <p>State or imply valid method for conclusion e.g.: Reject <math>H_0</math> if <math> r  &gt;</math> tab. value (AEF) <b>M1</b></p> <p>Correct conclusion (AEF, dep A1*, B1*): There is non-zero correlation <b>A1</b></p>	6	11
10A	<p>Find MI of lamina about <math>Q</math>: <math>I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2</math> <b>M1 A1</b>  <math>[= (15/4 + 81/4) ma^2 = 24 ma^2]</math></p> <p>State or find MI of rod about <math>Q</math>: <math>I_{\text{rod}} = (\frac{1}{3} + 1) M(3a/2)^2</math> <math>[= 3Ma^2]</math> <b>B1</b></p> <p>Sum to find MI of object about <math>Q</math>: <math>I_1 = 24 ma^2 + 3 Ma^2</math>  <math>= 3(8m + M) a^2</math> <b>A.G. A1</b></p> <p>Find MI of object about mid-point of <math>PQ</math>: <math>I_2 = (15/4 + 3^2) ma^2 + \frac{1}{3} M(3a/2)^2</math>  <math>= (51/4) ma^2 + \frac{3}{4} Ma^2</math>  <math>= \frac{3}{4} (17m + M) a^2</math> <b>A.G. M1 A1</b></p> <p>Use eqn of circular motion to find <math>d^2\theta/dt^2</math> for axis <math>l_1</math>:  <math>[-]I_1 d^2\theta/dt^2 = mg \times (9a/2) \sin \theta + Mg \times (3a/2) \sin \theta</math> <b>M1 A1</b>  <math>[= (9m/2 + 3M/2) ga \sin \theta]</math></p> <p>[Approximate <math>\sin \theta</math> by <math>\theta</math> and] find <math>\omega_1^2</math> in SHM eqn:  <math>\omega_1^2 = (3m + M)g / 2(8m + M) a</math> <b>M1</b></p> <p>Find period <math>T_1</math> for axis <math>l_1</math> from <math>2\pi/\omega_1</math>: (AEF) <math>T_1 = 2\pi\sqrt{\{2(8m + M) a / (3m + M)g\}}</math> <b>A1</b></p> <p>Use eqn of circular motion to find <math>d^2\theta/dt^2</math> for axis <math>l_2</math>:  <math>[-]I_2 d^2\theta/dt^2 = mg \times 3a \sin \theta</math> <b>M1</b></p> <p>[Approximate <math>\sin \theta</math> by <math>\theta</math> and] find <math>\omega_2^2</math> in SHM eqn:  <math>\omega_2^2 = 4mg / (17m + M) a</math> <b>M1</b></p> <p>Find period <math>T_2</math> for axis <math>l_2</math> from <math>2\pi/\omega_2</math>: (AEF) <math>T_2 = 2\pi\sqrt{\{(17m + M) a / 4mg\}}</math> <b>A1</b></p> <p>Verify that <math>T_1 = T_2</math> when <math>m = M</math>: (AEF) <math>T_1 = 2\pi\sqrt{(18 a / 4g)} = T_2</math> <b>B1</b></p> <p>[Taking <math>m = M</math> throughout 2<sup>nd</sup> part can earn:  M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]</p>	4  2	14

Page 8	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2015	9231	23

Question Number	Mark Scheme Details	Part Mark	Total Mark
<b>10B</b>	State hypotheses (B0 for $\bar{x}$ ...), e.g.: $H_0: \mu_X = \mu_Y, H_1: \mu_X \neq \mu_Y$	<b>B1</b>	9
	State assumption (AEF): Distributions have equal variances	<b>B1</b>	
	Find sample means <u>and</u> estimate popln. variances: $\bar{x} = 4.2, \bar{y} = 4.8$ (allow biased here: 0.36 or 0.6 <sup>2</sup> ) $s_X^2 = (180 - 42^2/10) / 9 = 0.4$ or 0.6325 <sup>2</sup>		
	(allow biased here: 0.4183 or 0.6468 <sup>2</sup> ) $s_Y^2 = (281.5 - 57.6^2/12) / 11 = 0.4564$ or 251/550 or 0.6755 <sup>2</sup>	<b>M1</b>	
	Estimate (pooled) common variance: (note $s_X^2$ and $s_Y^2$ not needed explicitly) $s^2 = (9s_X^2 + 11s_Y^2) / 20$ (AEF) or $(180 - 42^2/10 + 281.5 - 57.6^2/12) / 20 = 0.431$ or 0.6565 <sup>2</sup>	<b>M1 A1</b>	
	Calculate value of $t$ (to 3 s.f.): [-] $t = (\bar{y} - \bar{x}) / s \sqrt{(1/10 + 1/12)} = 2.13$	<b>M1 A1</b>	
	State or use correct tabular $t$ value: (or can compare $\bar{y} - \bar{x} = 0.6$ with 0.586) $t_{20, 0.975} = 2.086$ [allow 2.09]	<b>B1*</b>	
	Correct conclusion (AEF, $\sqrt{\quad}$ on $t$ , dep *B1): <b>S.R.</b> Implicitly taking $s_X^2, s_Y^2$ as popln. variances: (may also earn first B1 B1 M1) $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)} = 0.6 / \sqrt{0.078} = 2.15$	<b>B1<sup>✓</sup></b>	
	Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9) $2.15 > 1.96$ so mean masses not same	<b>(B1)</b>	
	State hypotheses (B0 for $\bar{x}$ ...), e.g.: $H_0: \mu_X = 3.8, H_1: \mu_X > 3.8$ or $H_0: \mu_X = \mu_Z, H_1: \mu_X > \mu_Z$	<b>B1</b>	
	Calculate value of $t$ using $s_X$ from above: $t = (4.2 - 3.8) / (s_X / \sqrt{10}) = 2.0$	<b>M1 A1</b>	
	State or use correct tabular $t$ value: (or can compare 0.4 with 0.367) $t_{9, 0.95} = 1.833$ [allow 1.83]	<b>B1*</b>	
Correct conclusion (A.E.F., $\sqrt{\quad}$ on $t$ , dep *B1): $t > 1.833$ , so claim is justified or mean mass of Royals > mean mass of Crowns	<b>B1<sup>✓</sup></b>	5	<b>14</b>