CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the October/November 2015 series

9231 FURTHER MATHEMATICS

9231/21 Paper 2, maximum raw mark 100

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CAMBRIDGE
International Examinations

Question Number	Mark Scheme Details	Part Mark	T[otal]
1	Find 3 independent equations for T, R_A, R_B: Resolve horizontally: $R_B = T\cos\alpha$ **M1 A1** Resolve vertically: $R_A = W + T\sin\alpha$ **M1 A1** Take moments about A: (*a* may be omitted from moment eqns) $R_B\,3a\sin\theta = W(3a/2)\cos\theta$ $+ T a(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$ *or* $+ T a\sin(\alpha+\theta)$ *or* $+ T\,3a\cos\theta\sin\alpha$ **M1 A1** Take moments about B: $R_A 3a\cos\theta = W(3a/2)\cos\theta$ $+ T\,2a(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$ *or* $+ T\,2a\sin(\alpha+\theta)$ *or* $+ T\,3a\sin\theta\cos\alpha$ **(M1 A1)** Take moments about C: $R_A\,a\cos\theta + W(a/2)\cos\theta = R_B\,2a\sin\theta$ **(M1 A1)** Take moments about D: $R_A\,3a\cos\theta - W(3a/2)\cos\theta = R_B\,3a\sin\theta$ **(M1 A1)** Solve for T, R_A, R_B (AEF in W and α): $T = W/2\sin\alpha$ *or* $\frac{1}{2}W\operatorname{cosec}\alpha$ **B1**; $R_A = 3W/2$ **B1**; $R_B = W/2\tan\alpha$ *or* $\frac{1}{2}W\cot\alpha$ **B1**	9	**9**
2	For A & B use conservation of momentum, e.g.: $2mv_A + mv_B = 2mu$ (allow $2v_A + v_B = 2u$) **M1** Use Newton's law of restitution (consistent signs): $v_B - v_A = eu$ **M1** Combine to find v_A and v_B: $v_A = (2-e)u/3$, $v_B = 2(1+e)u/3$ **A1, A1**	4	
	Find e from $v_A = \lvert v_B' \rvert$ with $v_B' = [-]\,0{\cdot}4\,v_B$: $(2-e) = 0{\cdot}8(1+e)$, $e = 2/3$ **M1 A1**	2	
	EITHER: Equate times in terms of reqd. distance x: $(d-x)/v_A = d/v_B + x/v_B'$ (AEF) **M1 A1** [speeds need not be found: $v_A = v_B' = 4u/9$, $v_B = 10u/9$] Use $v_A = v_B' = 0{\cdot}4\,v_B$ to solve for x: $d - x = 0{\cdot}4\,d + x$, $x = 0{\cdot}3\,d$ **M1 A1** *OR:* Find dist. moved by A when B reaches wall: $d_A = (d/v_B)\,v_A = 0{\cdot}4\,d$ **(M1 A1)** Find reqd. distance x: $x = \frac{1}{2}(d - d_A) = 0{\cdot}3\,d$ **(M1 A1)**	4	**10**

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3	Find k by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg\,(a/2)/a = 2mg\,(3a/2 - ka) / ka$ $½ = 3/k - 2, \quad k = 6/5$ *or* $1{\cdot}2$	**M1 A1** **A1**	3	
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m\,\mathrm{d}^2x/\mathrm{d}t^2 = -mg\,(a/2 + x)/a$ $+ 2mg\,(3a/2 - ka - x) / ka$ *or* $m\,\mathrm{d}^2y/\mathrm{d}t^2 = +mg\,(a/2 - y)/a$ $- 2mg\,(3a/2 - ka + y) / ka$	**M1 A2**		
	Simplify to give standard SHM eqn, e.g.: **S.R.**: B1 if no derivation (max 2/5)	$\mathrm{d}^2x/\mathrm{d}t^2 = -(1 + 2/k)gx / a$ $= -8gx/3a$	**A1**		
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: ($\sqrt{}$ on ω)	$T = 2\pi\sqrt{(3a/8g)}$ *or* $\pi\sqrt{(3a/2g)}$ *or* $3{\cdot}85\sqrt{(a/g)}$ *or* $1{\cdot}22\sqrt{a}$ [s]	**B1**$\sqrt{}$	5	
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$:	$0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$	**M1 A1**		
	Solve to find numerical value of a:	$0{\cdot}49 = (8g/3) \times 0{\cdot}0375a, \quad a = 0{\cdot}49$	**A1**	3	**11**

Question Number	Mark Scheme Details			Part Mark	T
4	*EITHER:* Find tension at top from $F = ma$ vertically:	$T = mu^2/a - mg$	B1		
	OR: Use energy at e.g. θ to upward vertical:	$\tfrac{1}{2} mv^2 = \tfrac{1}{2} mu^2 + mga(1 - \cos\theta)$			
	Find tension T by using $F = ma$ radially:	$T' = mv^2/a - mg\cos\theta$			
	Eliminate v^2:	$= mu^2/a + mg(2 - 3\cos\theta)$			
	Find T at top by taking $\theta = 0$:	$T = mu^2/a - mg$	(B1)		
	Find u_{min} by requiring $T \geqslant 0$ at top [or $T > 0$]:	$u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$ A.G.	B1	2	
	Find v at bottom from conservation of energy:	$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + mg \times 2a$	M1		
		$v^2 = ag + 4ag,\ v = \sqrt{(5ag)}$	A1		
	Find new speed V from conservation of momentum:	$m'V = mv$ with $m' = m + \tfrac{1}{4}m$	M1		
		$V = 4v/5 = 4\sqrt{(ag/5)}$ *or* $(4/5)\sqrt{(5ag)}$ AEF	A1	4	
	Find w^2 at angle θ from conservation of energy: (condone m instead of m' here since cancels out)	$\tfrac{1}{2} m'w^2 = \tfrac{1}{2} m'V^2 - m'ga(1 + \cos\theta)$	M1 A1		
		$[\,w^2 = ag(6/5 - 2\cos\theta)]$			
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3\cos\theta)$]	$\tfrac{1}{2} m'w^2 = \tfrac{1}{2} mu^2 + mga(1 - \cos\theta) - \tfrac{1}{4}mga(1 + \cos\theta)$	(B1)		
	Find tension T' there by using $F = ma$ radially:	$T' = m'w^2/a - m'g\cos\theta$	B1		
	Eliminate w^2:	$= m'V^2/a - m'g(2 + 3\cos\theta)$	A1		
	Substitute for m' and V: AEF	$= (5mg/4)(6/5 - 3\cos\theta)$ *or* $3mg/2 - (15/4)mg\cos\theta$	A1	5	
	Find $\cos\theta$ when string becomes slack from $T' = 0$: **S.R.** Allow if found from $T' = mg(6/5 - 3\cos\theta)$	$\cos\theta = \tfrac{1}{3} \times 6/5 = 2/5$ *or* 0·4	M1 A1	2	13
5	Find or use sample mean	$\bar{x} = 222{\cdot}8 / 10 = 22{\cdot}28$			
	<u>and</u> estimate population variance: (allow biased here: 0·412 *or* $0{\cdot}642^2$)	$s^2 = 4{\cdot}12 / 9 = 0{\cdot}458$ *or* 103/225 *or* $0{\cdot}677^2$	M1		
	Find confidence interval (allow z in place of t) e.g.:	$22{\cdot}28 \pm t\sqrt{(0{\cdot}458 / 10)}$	M1 A1		
	Use of correct tabular value:	$t_{9,\,0.975} = 2{\cdot}26[2]$	A1		
	Evaluate C.I. correct to 3 s.f.:	$22{\cdot}3 \pm 0{\cdot}48[4]$ *or* $[21{\cdot}8, 22{\cdot}8]$	A1	5	5

Question Number	Mark Scheme Details			Part Mark	T
6	Find prob. p of head from mean = 2 × variance:	$1/p = 2 \times (1-p)/p^2$, $p = ⅔$ **A.G.**	**M1 A1**	2	
(i)	Find P(X = 4) (denoting $1 - p$ by q [= ⅓]):	$P(X = 4) = q^3 \times p$ $= 2/81$ *or* $0{\cdot}0247$	**B1**	1	
(ii)	Find or state P(X > 4):	$P(X > 4)$ $[= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)\ ] = q^4$ $= 1/81$ *or* $0{\cdot}0123$	**M1 A1**	2	
(iii)	Formulate condition for N: Take logs (any base) to give bound for N: Find N_{min}: ($N < 6{\cdot}29$ *or* $N = 6{\cdot}29$ earns M2 A0)	$1 - q^N > 0{\cdot}999$, $[\ (⅓)^N < 0{\cdot}001\ ]$ $N > \log 0{\cdot}001 / \log ⅓$ $N > 6{\cdot}29$, $N_{min} = 7$	**M1** **M1** **A1**	3	**8**
7	Find F(x) for $1 \leqslant x \leqslant 4$:	$F(x) = (x^3 - 1)/63$	**B1**		
	Find G(y) from $Y = X^2$ for $1 \leqslant x \leqslant 4$: (result may be stated)	$G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$	**M1 A1**		
	Find g(y) for corresponding range of y:	$g(y) = y^{1/2}/42$ **A.G.**	**A1**		
	Find or state corresponding range of y :	$1 \leqslant y \leqslant 16$ **A.G.**	**B1**	5	
(i)	Find median value m of Y:	$(m^{3/2} - 1)/63 = ½$ $m = 32{\cdot}5^{\,2/3} = 10{\cdot}2$	**M1 A1**	2	
(ii)	Find E(Y) [or equivalently E(X^2)]:	$E(Y) = \int y\, g(y)\, dy = \int y^{3/2}\, dy / 42$ $= [y^{5/2}]_1^{16} / 105 = 1023/105$ $= 341/35$ *or* $9{\cdot}74$	**M1 A1**	2	**9**

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8	Find mean of sample data [for use in Poisson distn.]:	$\lambda = 220/100 = 2{\cdot}2$	B1		
	State (at least) null hypothesis (AEF):	H_0: Poisson distn. fits data *or* $\lambda = 2{\cdot}2$	B1		
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): (ignore incorrect final value here for M1)	11·080 24·377 26·814 19·664 10·8151 4·759 2·491	M1 A1		
	Combine last two cells so that exp. value ⩾ 5:	O_i: 3 E_i: 7·25	M1*		
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): (allow 7·95 if 1 d.p. exp.values used)	$\chi^2 = 0{\cdot}076 + 2{\cdot}879 + 0{\cdot}653 + 1{\cdot}448 + 0{\cdot}441 + 2{\cdot}491 = 7{\cdot}99$	M1 A1		
	State or use consistent tabular value (to 3 s.f.):	5 cells: $\chi_{3,0.95}{}^2 = 7{\cdot}815$ 6 cells: $\chi_{4,0.95}{}^2 = 9{\cdot}488$ (correct) 7 cells: $\chi_{5,0.95}{}^2 = 11{\cdot}07$	B1		
	State or imply valid method for conclusion e.g.:	Accept H_0 if χ^2 < tabular value	M1		
	Conclusion (AEF, requires both values correct): Not combining cells [so $\chi^2 = 8{\cdot}64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)	Distn fits *or* $\lambda = 2{\cdot}2$	A1	10	**10**
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$:	$S_{xy} = 24\,879 - 472 \times 400/8 = 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0{\cdot}6085$ (3 s.f.)	M1 A1		
	Find regression line of y on x:	$y = 400/8 + b_1 (x - 472/8) = 50 + 0{\cdot}6085 (x - 59) = 0{\cdot}6085x + 14{\cdot}1$	M1 A1		
	Find y when $x = 72$:	$= 57{\cdot}9$ *or* 58			
	Allow use of regression line of x on y (since neither variable clearly independent):	$S_{yy} = 21\,226 - 400^2/8 = 1\,226$			
		$b_2 = S_{xy} / S_{yy} = 1{\cdot}043$	(M1 A1)		
		$x = 472/8 + b_2 (y - 400/8) = 1{\cdot}043\,y + 6{\cdot}85$	(M1 A1)		
		$Y = 62{\cdot}5$ *or* 62	(A1)	5	

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	Find product moment correlation coefficient r:	$r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ *or* $\sqrt{(0{\cdot}6085 \times 1{\cdot}043)} = 0{\cdot}797$	M1 A1*		
	State both hypotheses (B0 for r …):	H_0: $\rho = 0$, H_1: $\rho \neq 0$	B1		
	State or use correct tabular two-tail r-value:	$r_{8,\,5\%} = 0{\cdot}707$	B1*		
	State or imply valid method for conclusion e.g.:	Reject H_0 if $\lvert r\rvert$ > tab. value (AEF)	M1		
	Correct conclusion (AEF, dep A1*, B1*):	There is non-zero correlation	A1	6	**11**
10A	Find MI of lamina about Q:	$I_{lamina} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\}$ $+ m(9a/2)^2$ $[= (15/4 + 81/4)\,ma^2 = 24\,ma^2]$	M1 A1		
	State or find MI of rod about Q:	$I_{rod} = (\frac{1}{3} + 1)\,M(3a/2)^2$ $[= 3Ma^2]$	B1		
	Sum to find MI of object about Q:	$I_1 = 24\,ma^2 + 3\,Ma^2$ $= 3\,(8m + M)\,a^2$ **A.G.**	A1	4	
	Find MI of object about mid-point of PQ:	$I_2 = (15/4 + 3^2)\,ma^2 + \frac{1}{3}M(3a/2)^2$ $= (51/4)\,ma^2 + \frac{3}{4}\,Ma^2$ $= \frac{3}{4}(17m + M)\,a^2$ **A.G.**	M1 A1	2	
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1:	$[-]I_1\, d^2\theta/dt^2 = mg \times (9a/2)\sin\theta$ $+ Mg \times (3a/2)\sin\theta$ $[= (9m/2 + 3M/2)\,ga\sin\theta\,]$	M1 A1		
	[Approximate $\sin\theta$ by θ and] find ω_1^2 in SHM eqn:	$\omega_1^2 = (3m + M)g / 2(8m + M)\,a$	M1		
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: (AEF)	$T_1 = 2\pi\sqrt{\{2(8m + M)\,a / (3m + M)g\}}$	A1		
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2:	$[-]I_2\, d^2\theta/dt^2 = mg \times 3a\sin\theta$	M1		
	[Approximate $\sin\theta$ by θ and] find ω_2^2 in SHM eqn:	$\omega_2^2 = 4mg / (17m + M)\,a$	M1		
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: (AEF)	$T_2 = 2\pi\sqrt{\{(17m + M)\,a / 4mg\}}$	A1		
	Verify that $T_1 = T_2$ when $m = M$: (AEF) [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	$T_1 = 2\pi\sqrt{(18\,a / 4g)} = T_2$	B1	8	**14**

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10B	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = \mu_Y$, H_1: $\mu_X \neq \mu_Y$	B1		
	State assumption (AEF):	Distributions have equal variances	B1		
	Find sample means and estimate popln. variances: (allow biased here: $0{\cdot}36$ *or* $0{\cdot}6^2$) (allow biased here: $0{\cdot}4183$ *or* $0{\cdot}6468^2$)	$\bar{x} = 4{\cdot}2$, $\bar{y} = 4{\cdot}8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0{\cdot}4$ *or* $0{\cdot}6325^2$ $s_Y^2 = (281{\cdot}5 - 57{\cdot}6^2/12) / 11$ $= 0{\cdot}4564$ *or* $251/550$ *or* $0{\cdot}6755^2$	M1		
	Estimate (pooled) common variance: (note s_X^2 and s_Y^2 not needed explicitly)	$s^2 = (9\,s_X^2 + 11\,s_Y^2) / 20$ (AEF) *or* $(180 - 42^2/10 + 281{\cdot}5 - 57{\cdot}6^2/12) / 20$ $= 0{\cdot}431$ *or* $0{\cdot}6565^2$	M1 A1		
	Calculate value of t (to 3 s.f.):	[-] $t = (\bar{y} - \bar{x}) / s\,\sqrt{(1/10 + 1/12)}$ $= 2{\cdot}13$	M1 A1		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0{\cdot}6$ with $0{\cdot}586$)	$t_{20,\,0.975} = 2{\cdot}086$ [allow $2{\cdot}09$]	B1*		
	Correct conclusion (AEF, √ on t, dep *B1): **S.R.** Implicitly taking s_X^2, s_Y^2 as popln. variances: (may also earn first B1 B1 M1) Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9)	$t > 2{\cdot}09$ so mean masses not same $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0{\cdot}6 / \sqrt{(0{\cdot}078} = 2{\cdot}15$ $2{\cdot}15 > 1{\cdot}96$ so mean masses not same	B1√ (B1)	9	
	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = 3{\cdot}8$, H_1: $\mu_X > 3{\cdot}8$ *or* H_0: $\mu_X = \mu_Z$, H_1: $\mu_X > \mu_Z$	B1		
	Calculate value of t using s_X from above:	$t = (4{\cdot}2 - 3{\cdot}8) / (s_X/\sqrt{10}) = 2{\cdot}0$	M1 A1		
	State or use correct tabular t value: (or can compare $0{\cdot}4$ with $0{\cdot}367$)	$t_{9,\,0.95} = 1{\cdot}833$ [allow $1{\cdot}83$]	B1*		
	Correct conclusion (A.E.F., √ on t, dep *B1):	$t > 1{\cdot}833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns	B1√	5	14