FURTHER MATHEMATICS

Paper 9231/01

Paper 1

General comments

The majority of candidates produced good work in response to over 10 questions. There were very few misreads and the rubric infringement of attempting both options of **Question 12** appeared in only a small minority of scripts. Some candidates may have run into time trouble with the final question but the general impression given was that they did all that they could do and that loss of marks was due mainly to elementary errors in the working.

The standard of presentation of work was very Centre dependent. At one extreme, Centres produced large amounts of ideal presentation which was easy to read and very well organised. At the other, there were badly organised and scarcely legible responses. Nevertheless standards in this respect have improved relative to that of last year, and no doubt this aspect of candidate work goes some way to explaining why, this year, there were fewer substandard scripts.

Topics which appeared to be only partially understood, at least in some Centres, were induction, curve sketching, both in cartesian and polar coordinates, three dimensional vector problems, implicit differentiation up to order 2 and limiting processes. Nevertheless, apart from these, the remaining syllabus topics which featured in this paper appear to have been very well understood.

Comments on Individual Questions

Question 1

The majority of candidates answered this question correctly. They took the most direct route in that they could see at once that the eigenvalues could be obtained simply by reading off the elements of the leading diagonal. Such a procedure is not valid in general, but is valid in situations where all the elements below (or above) the leading diagonal are zero. In contrast, there were those who first obtained the characteristic equation and then solved for the eigenvalues, λ. This unnecessary complication, usually accurately resolved, must have wasted large amounts of examination time.

The obtaining of the eigenvectors generated some suboptimal responses in that laborious and badly organised work proliferated. A simple method here, adopted by a minority, is to evaluate, for each value of λ, the vector product of any 2 rows of the matrix $\mathbf{A} - \lambda\mathbf{I}$. In this respect, a common error was to evaluate vector products of columns.

Answers: 1, 2, –3; $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}9\\-4\\20\end{pmatrix}$.

Question 2

Many responses were marred by elementary errors such as one would not expect to see at this level. The majority of candidates followed the suggested method, that is they first obtained $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{n+1}\mathrm{e}^{-x^3}\right) = (n+1)x^n\mathrm{e}^{-x^3} - 3x^{n+3}\mathrm{e}^{-x^3}$ and then integrated with respect to x over the interval $[0, 1]$. Provided careful attention is paid to detail, the required result follows easily enough, but as it was, even this simple strategy generated errors. The alternative strategy of integrating by parts was adopted by a minority but again, errors of the most elementary kind precluded a valid argument. Moreover, in both strategies, the limits of integration were not always specified at each stage of the argument.

The second part of the question only requires elementary algebra for the obtaining of I_6 in terms of e and I_0: No calculus, as such, is involved. Nevertheless, a minority of candidates obtained the required result. Some candidates understood the question to mean that I_0 must first be evaluated in terms of e even though this is not possible.

Answer: $I_6 = \frac{4}{9}I_0 - \frac{7}{9}\mathrm{e}^{-1}$.

Question 3

In contrast to the previous question, the working here was generally accurate. Very few candidates failed to make some progress.

Most responses showed about the right amount of detail to establish the first result.

For the rest of the question, it was generally understood that the method of differences based on the first result was involved, so that most candidates obtained $\sum_{n=1}^{N} u_n = \frac{v_{N+1} - v_1}{m+1}$ However, a minority of candidates were unable to translate this expression into a correct result in terms of m and N, such as the one given below.

Answer: $\frac{(N+1)(N+2)...(N+1+m)}{m+1} - m!$

Question 4

The majority of responses showed a statement of, or at least implied, a correct inductive hypothesis. H_n. In contrast, a minority of candidates began by identifying H_n with a statement of the question, so indicating a complete misunderstanding of the principle of mathematical induction. This fundamental error has occurred in responses to questions on induction in previous examinations of this syllabus and comment on it has been made in corresponding reports.

The essence of the proof, which requires showing that $7|(10^{3k} + 13^{k+1}) \Rightarrow 7|(10^{3k+3} + 13^{k+2})$ was established by most candidates, even if they had failed to define H_n. In this respect, one must remark that some of the working at this stage was complicated, to say the least, and it is therefore much to the credit of some candidates that they managed to find their way through some very obscure detail.

Finally, the majority of responses showed a satisfactory conclusion to the induction argument. Very few failed to make clear the range of n for which the divisibility property is valid.

Question 5

Most candidates had a clear idea of what was expected of them. The most popular strategy was to attempt to reduce the matrix

$$\begin{pmatrix} 2 & 3 & 4 & -5 \\ 4 & 5 & -1 & 5a+15 \\ 6 & 8 & a & b-2a+21 \end{pmatrix}$$

to an echelon form such as

$$\begin{pmatrix} 2 & 3 & 4 & -5 \\ 0 & 1 & 9 & -5a-25 \\ 0 & 0 & a-3 & b-7a+11 \end{pmatrix}.$$

From here both parts of the question can be answered immediately. However, there were many arithmetic errors in the working so that complete success was achieved only by a minority.

Those who worked with equations did less well mainly because their algebra was badly organised. The syllabus does not demand a knowledge of the concept of the echelon form but nevertheless, it is clear that its application to problems of this type is more likely to lead to success than the undisciplined implementation of Guassian elimination.

Answer: 10.

Question 6

The formidable appearance of this question did not deter the majority of candidates from finding a simple solution, as suggested by the question itself. One could say that this was the best answered question of the paper.

Most responses began by using the inverse of the relation $y = \dfrac{4x+1}{x+1}$, namely $x = \dfrac{y-1}{4-y}$ to transform the given equation into a cubic in *y,* and for the most part the working was accurate.

The majority wrote S_n for the sum of the *n*th powers of the roots of the new equation (a helpful notational simplification) and proved that $S_2 = -2p$ and also that $S_3 + pS_1 + 3q = 0$ from which the required results can be obtained immediately.

In contrast, a significant minority attempted to express S_2 and even S_3 in terms of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, where α, β and γ are the roots of the given cubic equation. Such an error prone strategy, which was seldom implemented with success, must have used up a lot of examination time.

Answers: $p = -21$, $q = 47$; 42, -141.

Question 7

The standard of responses to this question was high, especially in the second part.

The general aspect of the required graph was generaly correct in outline. However, nearly half of all the sketches presented did not show the line $\theta = 0$ to be a tangent to *C* at the pole.

The evaluation of the area started in nearly all cases with the correct integral representation. The transformation to the *w*-domain was usually effected accurately, as was the subsequent integration. Few failed to obtain $(w^2 - 2w + 2)e^w$ as an integral of w^2e^w. Nevertheless, the limits of integration were not always made clear and sometimes they were omitted altogether. Moreover, some candidates did not heed the advice of several previous reports, namely that where the answer is given it is essential that all relevant and necessary working is presented.

Question 8

In the first part of the question about half of all candidates started with $v_1 = 4y^3y_1$ together with $v_2 = 4y^3y_2 + 12y^2y_1^2$ $(*)$ where $v_n = \dfrac{d^nv}{dx^n}$, $y_n = \dfrac{d^ny}{dx^n}$. From these results the given *x*-*y* equation can easily be worked into the required x_v form. Such a strategy is completely undermined if $v_2 = 4y^3y_2 + 12y^2y_1$ is used in place of (*) yet this error, early on in the working, appeared in a substantial minority of scripts.

The rest of the candidature who attempted to obtain results for y_1 and y_2 based on $y = v^{1/4}$ did markedly less well. The fractional indices caused difficulties for many and some erroneous working was deliberately distorted so as to lead to the new differential equation.

Almost all responses showed an essentially correct strategy for the obtaining of the solution of the x-v differential equation. The most persistent errors were the incorrect solution of the auxiliary quadratic equation and the failure to translate the solution for v into the solution for y.

Answer: $y = [e^{-3x}[A\sin(5x) + B\cos(5x)] + e^{-4x}]^{1/4}$.

Question 9

Many responses began by showing that the direction of the common perpendicular of the lines AB and OC is parallel to the vector $2\mathbf{i} - \mathbf{k}$; but subsequently made little progress with the first part of this question. In fact, there are 2 possible strategies here. For the first it is sufficient to write $\mathbf{i} + \lambda\mathbf{j} + \mu(2\mathbf{i} - \mathbf{k}) = \nu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and then to solve for λ; μ and ν by comparing the coefficients of the basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

The second is to write down orthogonality conditions such as $\overrightarrow{PQ}.\mathbf{j}$ and $\overrightarrow{PQ}.(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 0$, where P and Q are general points on AB and OC, respectively, and then to solve for parameter values. Generally, the second strategy was preferred but in both cases failure was usually due to elementary errors in the working.

The determination of the shortest distance, d, between the lines AB and OC was carried out accurately by most candidates. Here the most popular and certainly the least error prone strategy was based on the evaluation of a preliminary result such as $\dfrac{|\mathbf{i}.(2\mathbf{i} - \mathbf{k})|}{\sqrt{5}}$ Others, however, used the results from their orthogonality conditions to evaluate d.

The last part of this question also went well. The geometry was understood by the majority of candidates so that vector evaluations were generally relevant to the result required. Thus in most responses something essentially like $\mathbf{j} \times (2\mathbf{i} - \mathbf{k}) = -\mathbf{i} - 2\mathbf{k}$ appeared from which the required cartesian equation follows almost immediately.

Answers: $\mathbf{r} = \mathbf{i} + \frac{1}{5}\mathbf{j} + \mu(2\mathbf{i} - \mathbf{k})$; $x + 2z = 1$.

Question 10

Most candidates had some difficulty with this question. The main reasons for the poor quality of many responses were notational obscurities, elementary errors in the working and lack of explanatory skills. No doubt candidates generally had previously worked through many problems on implicit differentiation of a purely routine nature. In contrast this question went beyond such standardisation in that it drew on a wide range of skills that come within the boundaries of the syllabus.

In the first place, all properly prepared candidates should at least have been able to twice differentiate the given equation with respect to x. In this respect it is optimal, in terms of examination time, to differentiate directly without first carrying out any preliminary rearrangement. The implementation of this strategy will lead to $y_1 = 2x + \lambda(1 + y_1)\cos(x + y)$ (*) and $y_2 = 2 + \lambda y_2\cos(x + y) - \lambda(1 + y_1)^2\sin(x + y)$ (**), where $y_n = \dfrac{d^n y}{dx^n}$.

In fact many candidates deviated from this approach in that (*) was rearranged so as to make y_1 the subject before the second differentiation was attempted. This led to results which were a lot more complex than (**) and very often wrong.

Once the formal differentiation is complete it is then possible to consider the remaining aspects of the question. Given that the curve passes through $A(\frac{\pi}{4}, \frac{\pi}{4})$ it follows that $\lambda = \frac{\pi}{4} - \frac{\pi^2}{16}$, a result obtained by most candidates. Rearrangement of (*) will show that y_1 can only become infinite if $\cos(x + y) = \frac{1}{\lambda}$ for some x, y. This is impossible since $\frac{1}{\lambda} > 1$.

For the final part of the question, it is first necessary to observe that (*) implies $y_1 = \frac{\pi}{2}$ *at A*. It is then a simple matter to obtain the required result by using this value together with $x = y = \frac{\pi}{4}$ in (**).

Question 11

Most candidates managed to make good progress with this question. Errors occurred mainly in the first and final parts of the question.

The majority of candidates proved, or attempted to prove de Moivre's theorem for a positive integral index by induction. The comments with regard to the inductive hypothesis made in this report for **Question 4** apply here. Moreover, the working for the central part of the proof where it is necessary to *prove* that $(\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta) = \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$ was deficient in a number of scripts.

Almost all candidates produced a complete and correct response to the second part of the question. The large amounts of detailed working on display were impressive and they provided evidence of a candidature well prepared for this type of problem.

For the concluding part of this question, there were many incomplete responses. Almost everyone got as far as showing that $\cos 7\theta = -\frac{1}{2}$ and hence that all the roots of the equation could be expressed in the required form by appropriate choice of θ. Nevertheless only about half of all candidates obtained exactly 7 distinct roots.

Answer: $\cos\left(\frac{2\pi}{21} + \frac{2k\pi}{7}\right)$, $k = 0, 1, \ldots, 6$.

Question 12 EITHER

(i) All candidates wrote down the correct equation of the vertical asymptote and by some valid method obtained an equation of the form $y = \frac{x}{2} + c$ for the diagonal asymptote. However, in some responses the constant c was not correctly identified.

(ii) Several complicated strategies featured here and a minority of candidates failed to obtain the correct value of q. All that was required was a statement to the effect that for tangency with the x-axis the discriminant of the quadratic form $x^2 + qx + 1$, that is $q^2 - 4$, must be zero. Since it is given that $q > 0$, then $q = 2$.

If q is incorrect then a completely correct sketch graph is unlikely to appear. Even among those whose value of q was correct there was a manifest lack of comprehension of what C looks like in this special situation. Widespread errors were bad forms at infinity, an incorrect left-hand branch, an incorrect point of contact with the x-axis by the right-hand branch. Only a minority of candidates produced a completely correct sketch graph for this part of the question.

(iii) Unexpectedly, responses here were generally of a better quality than those for part **(ii)**. Most sketch graphs exhibited a correct overall appearance. The intersections of C with the x-axis were usually identified correctly. Frequent errors were the failure to draw the diagonal asymptote through the point $(-\frac{3}{2}, 0)$ and for the intersection of the right-hand branch with the x-axis to be placed to the right of the origin.

(iv) There were very few completely correct reponses to the final part of this option. Some candidates attempted to argue by non-graphical methods, even though the question requires use of the diagrams already drawn. Among the minority who did draw representative lines through the point of intersection of the asymptotes very few could supply coherent explanations as to why their augmented diagrams validated the conclusions of the question.

Answers: **(i)** $x = -\frac{3}{2}$, $y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4}$; **(ii)** $q = 2$; **(iii)** $\left(\frac{-3-\sqrt{5}}{2}, 0\right)$, $\left(\frac{-3+\sqrt{5}}{2}, 0\right)$.

Question 12 OR

The correct evaluation of the first result appeared in most responses. There were very few cases of insufficient working appearing in order to obtain the displayed result.

In the middle section there were few incorrect integral representations of the required surface area, S, The actual evaluation of the correct definite integral generated much unnecessarily complicated working. The simple and most direct strategy, based on $S = \pi\int_0^3\left(x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + x^{1/2}\right)\mathrm{d}x + 2\lambda\pi \times 2\sqrt{3}$, appeared in very few responses.

Most of the survivors who got as far as this stage of the question evaluated $\int_0^3 y\,\mathrm{d}x$ correctly in terms of λ and so, using the result for $\int_0^3 y^2\,\mathrm{d}x$ given in the question, obtained a result for h. Almost without exception they went on to assemble an expression in terms of λ for $\frac{S}{hs}$ but then failed to establish the given limit. In fact all that was necessary was first to show in some intelligible notation that $h = \frac{\lambda}{2} + O(1)$, $\frac{S}{s} = 2\lambda\pi + O(1)$ and hence that $\frac{S}{hs} = 4\pi + O(\lambda^{-1})$. The required result is then immediate.

Answer: $S = 3\pi + 4\lambda\pi\sqrt{3}$.

FURTHER MATHEMATICS

Paper 9231/02

Paper 2

General comments

Since almost all candidates attempted the required number of questions, there seems as in previous years to have been no undue time pressure. Good attempts were seen at all questions, though **Questions 1** and **2** seemed to be found most challenging among the compulsory ones. In the single question which offers an alternative, namely **Question 11**, the Mechanics option proved more popular than has often been the case previously, and the given answers were regularly obtained, though the final part proved more difficult.

Comments on specific questions

Question 1

Since the motion here is rotational, the moment of inertia $\frac{10ma^2}{3}$ of the combined body must be found, using the parallel axes theorem for the rod, but many candidates failed to do so correctly or did not even attempt it. Even more used conservation of energy in an invalid attempt to find the angular acceleration. While this approach can lead to the desired result by differentiating, it requires that the change in potential energy be expressed in terms of a variable angle of rotation θ, whereas most such attempts had $\frac{1}{3}\pi$ in place of θ. A more sensible approach is to equate the product of the moment of inertia and the angular acceleration to the moment of the body about the hinge.

Answer: $\frac{3g}{10a}$.

Question 2

Like the previous one, this question was not particularly well answered. In essence it requires one energy equation for the downward motion and another for the upward motion, together with the fact that the speed immediately after hitting the table is e times the speed before. Common faults were to omit the elastic energy in the first or even more often the second equation, to attempt to use SHM while the string is extended but then introduce a constant acceleration, or to confuse signs.

Answer: $\frac{1}{3}$.

Question 3

The first part is most easily done by taking moments about B, though there are several equally valid ways of formulating the moment of W. Some candidates chose to effectively resolve W parallel to the sides of the block, but then included only one of these in their moment equation. The upper limit on α follows from applying $F \leq \mu R$ at B, with F and R found by resolving forces, and was often shown correctly. The lower limit was more problematic, producing a variety of invalid arguments such as $\mu \geq 0$, but can be shown by, for example, arguing that the expression for the force exerted by the wall at A must be non-negative or that (equivalently) the centre of the block must not be to the right of B.

Question 4

Most candidates found the given magnitude R of the force correctly, by noting that the initial speed is $\frac{I}{m}$ and then combining an energy equation and a radial resolution of forces. An upper limit for I follows from $R \geq 0$ when $\theta = \pi$, but most candidates overlooked the possibility that I might be so small that the car does not reach a higher level than O. This corresponds to the requirement that the expression for its speed should be no greater than zero when $\theta = \frac{1}{2}\pi$.

Answer: $I \leq m\sqrt{(2ga)}$, $I \geq m\sqrt{(5ga)}$.

Question 5

A more successful approach is to first determine the equilibrium point, and then find an equation of motion involving the displacement from this point since this leads immediately to the standard form of the SHM equation with ω^2 here $\frac{4g}{a}$. The period T may then be found from the usual formula $\frac{2\pi}{\omega}$. The alternative approach of measuring the displacement from some other point leads to an additional constant term in the equation of motion, and the origin should really be changed (effectively to the equilibrium position) so as to transform the equation into the standard SHM form. While a number of candidates found the lower condition on l correctly, many wrongly took the upper limit as $3a$. The speed at which the string first becomes slack may be found either from the standard SHM formula $v^2 = \omega^2(A^2 - x^2)$ with $A = \frac{3}{4}a$ and $x = \frac{1}{4}a$ or by equating kinetic and elastic energies. A common fault in the final part was to use $x = A \cos \omega t$ with $x = \frac{1}{4}a$ when it should be $-\frac{1}{4}a$.

Answers: **(i)** $2a < l < \frac{5}{2}a$; **(ii)(a)** $\sqrt{(2ga)}$, **(b)** $0.955\sqrt{\frac{a}{g}}$.

Question 6

The method for calculating an unbiased estimate of the population variance and hence the required confidence interval was widely understood, though some candidates used the critical value 1.960 from the normal distribution for the latter instead of the t-value 2.776. A complete answer to the second part would involve a consideration of the effect of the changed variance, critical t-value and sample size on the expression for the confidence interval, all of which tend to make it narrower, but few candidates considered more than one of these factors.

Answer: **(i)** [51.3, 56.5].

Question 7

The first part may be answered from the standpoint of either Tom missing ten successive shots, giving a probability 0.8^{10}, or the complement of his hitting the bull's-eye on any one of the first ten shots, which requires more calculation and is therefore a less attractive approach. The second part requires the summation over n of the probability $(0.8^{n-1} \times 0.2)^2$ of both Tom and Brad hitting the bull's-eye on their n th shot. The final part may be answered in an analogous fashion, a common fault being the inclusion of an extra factor 0.2, but it may also be found more easily as one half the complement of the total probability in the second part.

Answers: **(i)** 0.107; **(ii)** $\frac{1}{9}$; **(iii)** $\frac{4}{9}$.

Question 8

Most candidates stated the null and alternative hypotheses correctly, and went on to estimate either the variances of the two populations or a pooled variance, and hence a value of z or t of 1.56 (or 1.58 if biased estimates are used in the first approach). Comparison with the critical z-value of 2.326 or a nearby tabular t-value 2.358 or 2.390 leads to the conclusion that there is no difference in the mean times to extinction. The required answer to the final part, namely that the probability of concluding that the mean times are not equal when they are in fact equal is 2%, was not often seen. Instead many candidates made vague statements about 2% being the probability of error or of finding the mean times are unequal.

Question 9

After calculating the mean and variance, the majority of candidates concluded correctly that these being approximately equal supports the suggestion of a Poisson distribution, and usually went on to show that the expected value for $x = 3$ is 50 times the appropriate term in the distribution based on their calculated mean. While most candidates were familiar with the χ^2 test, not all combined the last three cells to give a χ^2 value of 5.38, or used the appropriate number of degrees of freedom. Comparison of the calculated χ^2 with the critical value 6.251 leads to the conclusion that a Poisson distribution fits the data.

Answers: 2.22, 2.13.

Question 10

The sum of the twelve values of y follows from either calculating the mean 0.95 of y from the given equation or equivalently summing both sides of this equation over the twelve observations, though some candidates who used the latter approach omitted to multiply 4.82 by 12. Rather than embark, as some did, on a lengthy and usually fruitless manipulation of the expressions for the given product moment correlation coefficient r in the *List of Formulae*, one need only recall that r^2 is the product of the given gradient –2.25 and the unknown gradient in the required regression line of x on y. Having found this gradient, the equation of the line then follows since the means of x and y are effectively known. Most candidates compared the given coefficient –0.3 with the critical value –0.497 to conclude that there is insufficient evidence of negative correlation, though a few wrongly compared the negative coefficient with 0.497 instead. The required value of y was usually estimated using the equation of the regression line of y on x, though the other line could be used. A valid comment would be that the lack of evidence of correlation makes this estimate unreliable, though many candidates instead argued on the basis of 2.8 being outside the range of observations even though this range is here unknown.

Answers: **(i)** 11.4; **(ii)** $x = 1.76 - 0.04y$; **(iv)** –1.48.

Question 11 EITHER

The Mechanics alternative proved more popular than has usually been the case in previous years, and perhaps aided by the given answers, the first two parts were often well done. The moment of inertia of the sign is found by using the expressions for the moments of inertia about their centres of the two identical discs and the rectangular lamina given in the *List of Formulae*, applying the parallel axes theorem, and combining the three results. The given expression for ω^2 is found by equating the changes in rotational and potential energy, considering either the three components separately or the complete sign for the latter. The final part proved more challenging, with relatively few candidates explaining that the assumption of AD becoming vertical is invalid, since the sign does not rotate that far. A variety of incorrect assertions were made, one of the more popular being that the support at F also breaks.

Question 11 OR

By contrast the Statistics option was less well answered on the whole. Some candidates began by stating the probability density function f(x) over the two segments as their answer to the first part, and even those who integrated these two expressions to find the required cumulative distribution function F(x) often omitted to add F(3) to their integral of f(x) over [3, x] when considering the interval $3 \le x \le 5$. The distribution function of Y may be found by substituting for the function F from the first part in $1 - F(5 - y^2)$, while in the final part $P(Y < 2X)$ may be shown to be successively $P((4X + 5)(X - 1) > 0)$, $P(X > 1)$ and hence $1 - F(1)$.

Answers: $\dfrac{x^2}{21}$ $(0 \le x \le 3)$, $\dfrac{2x-3}{7}$ $(3 < x \le 5)$, $\dfrac{2y^2}{7}$ $(0 \le y \le \sqrt{2})$, $\dfrac{-y^4+10y^2-4}{21}$ $(\sqrt{2} < y \le \sqrt{5})$; $\dfrac{20}{21}$.