CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the October/November 2013 series

9231 FURTHER MATHEMATICS

9231/12 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through $\sqrt{}$" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Mark	Total
1 (i)	Uses area formula.	Area $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4e^{2\theta}\,d\theta = \left[e^{2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$	M1		
	Obtains result.	$= e^{\pi} - e^{\frac{\pi}{3}} \quad (= 20.3)$	A1	2	
(ii)	Uses arc length formula.	Arc length $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4e^{2\theta} + 4e^{2\theta}}\,d\theta = 2\sqrt{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} e^{\theta}\,d\theta$	M1A1		
	Obtains result.	$= 2\sqrt{2}\left[e^{\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\sqrt{2}\left[e^{\frac{\pi}{2}} - e^{\frac{\pi}{6}}\right] \quad (= 8.83)$	A1	3	[5]
2 (i)	Finds S_2.	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ $= 0 - 2(-p) = 2p$ (AG)	B1	1	
(ii)	Finds S_3.	$\alpha^3 + \beta^3 + \gamma^3 = p\sum\alpha + 3q = 0 + 3q = 3q$ (AG)	M1A1	2	
(iii)	Finds S_5.	$\alpha^5 + \beta^5 + \gamma^5 = p\sum\alpha^3 + q\sum\alpha^2$ $= p.3q + q.2p = 5pq$ $\Rightarrow 6\sum\alpha^5 = 30pq = 5\sum\alpha^3\sum\alpha^2$ (AG)	M1 A1 A1	3	[6]
3	Writes first four sums.	$S_1 \ldots S_4 \sim 3, 10, 21, 36$	B1		
	Deduces first four terms, conjectures and justifies result.	$u_1 \ldots u_4 \sim 3, 7, 11, 15 \Rightarrow u_r = 4r - 1$ since $S_n = \frac{n}{2}\{6 + 4(n-1)\} = 2n^2 + n$ as given.	B1B1 B1		
		Or $u_r = S_r - S_{r-1} = 2r^2 + r - 2(r-1)^2 - (r-1)$ $= 4r - 1$	B1B1 B1	4	
	Obtains required sum.	$\sum_{n+1}^{2n}(4r-1) = 4.\frac{2n(2n+1)}{2} - 2n - \left(4.\frac{n(n+1)}{2} - n\right)$ $= 8n^2 + 2n - (2n^2 + n) = 6n^2 + n$	M1A1 A1	3	
		Or Sum of AP $= \frac{n}{2}(4n + 3 + 8n - 1) = 6n^2 + n$			[7]

Qu No	Commentary	Solution	Marks	Part Mark	Total
4	Integrates by parts.	$I_n\left[x^n(1+2x)^{\frac{1}{2}}\right]_0^1 - \int_0^1 \frac{nx^{n-1}(1+2x)}{(1+2x)^{\frac{1}{2}}}\mathrm{d}x$	M1A1		
		$=\sqrt{3} - n\int_0^1 \frac{x^{n-1}}{(1+2x)^{\frac{1}{2}}}\mathrm{d}x - 2n\int_0^1 \frac{x^n}{(1+2x)^{\frac{1}{2}}}\mathrm{d}x$			
	Obtains result.	$\Rightarrow (2n+1)I_n = \sqrt{3} - nI_{n-1}$ (AG)	A1	3	
	Alternatively:	$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^n(1+2x)^{\frac{1}{2}}\right\} = (1+2x)^{\frac{1}{2}}.nx^{n-1} + x^n(1+2x)^{-\frac{1}{2}}$	(M1)		
		$\Rightarrow \left[x^n(1+2x)^{\frac{1}{2}}\right]_0^1 = \frac{n(1+2x)x^{n-1}}{\sqrt{1+2x}} + I_n$	(A1)		
		$\Rightarrow (2n+1)I_n = \sqrt{3} - nI_{n-1}$ (AG)	(A1)		
	Finds I_0 (or I_1).	$I_0 = \left[\sqrt{1+2x}\right]_0^1 = \sqrt{3} - 1$	B1		
	Uses Red. Form.	$3I_1 = \sqrt{3} - (\sqrt{3} - 1) \Rightarrow I_1 = \frac{1}{3}$	M1		
	Finds I_2 and I_3.	$\Rightarrow I_2 = \frac{\sqrt{3}}{5} - \frac{2}{15} \Rightarrow I_3 = \frac{2}{35}(\sqrt{3}+1)$ (AG)	A1A1	4	[7]
5 (i)	Differentiates once,	$y' = 2(1+x)\ln(1+x) + (1+x)$	B1		
	twice	$y'' = 2\ln(1+x) + 3$	B1		
	and three times.	$y''' = \frac{2}{1+x}$	B1	3	
		(Allow B1√ if constant term in previous line incorrect.)			
(ii)	Proves base case.	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{(-1)^2.2.0!}{1+x} = \frac{2}{1+x} \Rightarrow \mathrm{H}_3$ is true.	B1		
	States inductive hypothesis.	$\mathrm{H}_k : \frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{(-1)^{k-1}.2.(k-3)!}{(1+x)^{k-2}}$ for some k.	B1		
	Differentiates	$\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = (-1)^{k-1}.2(k-3)!(-1)(k-2)(1+x)^{-(k-1)}$	M1		
	Proves inductive step and	$= \frac{(-1)^k.2.(k-2)!}{(1+x)^{k-1}} \Rightarrow \mathrm{H}_{k+1}$ is true	A1		
	states conclusion.	Hence by PMI H_n is true for all integers $\geqslant 3$	A1	5	[8]

Qu No	Commentary	Solution	Marks	Part Mark	Total
6	Reduces to echelon form.	$\begin{pmatrix}1 & -3 & -1 & 2\\ 4 & -10 & 0 & 2\\ 1 & -1 & 3 & -4\\ 5 & -12 & 1 & 1\end{pmatrix} \to \ldots \to \begin{pmatrix}1 & -3 & -1 & 2\\ 0 & 1 & 2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$	M1A1		
	Obtains rank	$r(\mathbf{M}) = 4 - 2 = 2$	A1		
		$\left.\begin{array}{r} x - 3y - z + 2t = 0 \\ y + 2z - 3t = 0 \end{array}\right\}$	M1		
		$\Rightarrow t = \mu, z = \lambda, y = 3\mu - 2\lambda, x = 7\mu - 5\lambda$	A1		
	and basis for null space.	Basis is : $\left\{\begin{pmatrix}-5\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}7\\3\\0\\1\end{pmatrix}\right\}$ (OE) e.g. $\begin{pmatrix}1\\0\\-3\\2\end{pmatrix}$ or $\begin{pmatrix}0\\1\\7\\5\end{pmatrix}$	A1	6	
	Evaluates matrix product.	$\mathbf{M}\begin{pmatrix}1\\-2\\-3\\-4\end{pmatrix} = \begin{pmatrix}2\\16\\10\\22\end{pmatrix}$	B1		
	Finds general solution of equations.	$\mathbf{x} - \begin{pmatrix}1\\-2\\-3\\-4\end{pmatrix} \in K$	M1		
		$\mathbf{x} = \begin{pmatrix}1\\-2\\-3\\-4\end{pmatrix} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ (AG)	A1	3	9
7	Proves first result.	$\mathbf{Ae} = \lambda\mathbf{e}$ $\Rightarrow \mathbf{A}^2\mathbf{e} = \mathbf{AAe} = \mathbf{A}\,\lambda\mathbf{e} = \lambda\,\mathbf{Ae} = \lambda^2\mathbf{e} \Rightarrow$ result. ($\mathbf{e} \neq \mathbf{0} \Rightarrow \lambda^2$ is an eigenvalue of $\mathbf{A}^2$.)	B1 M1A1	3	
	Obtains eigenvalues of **B**.	$(1-\lambda)(\lambda-4)(\lambda+2) = 0$ $\Rightarrow \lambda = -2, 1, 4$	M1A1 A1A1	4	
	Obtains eigenvalues of related matrix.	$1^4\mathbf{e} + 2\times 1^2\mathbf{e} + 3\mathbf{e} = 6\mathbf{e} \Rightarrow 6$ is an eigenvalue. $(-2)^4\mathbf{e} + 2\times(-2)^2\mathbf{e} + 3\mathbf{e} = 27\mathbf{e} \Rightarrow 27$ is an eigenvalue. $4^4\mathbf{e} + 2\times 4^2\mathbf{e} + 3\mathbf{e} = 291\mathbf{e} \Rightarrow 291$ is an eigenvalue.	M1A1 A1	3	10

Qu No	Commentary	Solution	Marks	Part Mark	Total
8	Finds normal to Π_1 .	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$	M1A1		
	Finds Cartesian equation.	Equation of Π_1 : $x + 3y - z = 12$	A1	3	
	Finds angle between normals, using scalar product.	$\cos\theta = \left\vert \dfrac{2-3-1}{\sqrt{11}\sqrt{6}} \right\vert$	M1		
		$= \dfrac{2}{\sqrt{66}} \Rightarrow \theta = 75.7°$ or 1.32 rad.	A1	2	
	Finds direction of line of intersection, using vector product.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$	M1A1		
	Finds point common to both planes.	Point on both planes is e.g. (6,2,0)	M1A1		
	States vector equation.	$\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$ (OE)	A1	5	[10]

9	Obtains area of surface of revolution.	$\dot{x} = 2t \quad , \quad \dot{y} = 1 - t^2$			
		$\Rightarrow \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = 4t^2 + 1 - 2t^2 + t^4 = \left(1 + t^2\right)^2$	M1A1		
		$2\pi \int y \, \mathrm{d}s = 2\pi \int_0^1 \left(t - \frac{1}{3}t^3\right)\left(1 + t^2\right)\mathrm{d}t$	M1A1		
		$= 2\pi \int_0^1 \left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5\right)\mathrm{d}t = 2\pi\left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^1$	M1		
		$= \frac{11}{9}\pi$ or 3.84	A1	6	
	Finds coordinates of centroid, using relevant formulae.	$\int y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = \int_0^1 \left(2t^2 - \frac{2}{3}t^4\right)\mathrm{d}t = \left[2\frac{t^3}{3} - \frac{2}{3}\frac{t^5}{5}\right]_0^1 = \frac{8}{15}$	B1		
		$\int xy \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = \int_0^1 \left(2t^4 - \frac{2}{3}t^6\right)\mathrm{d}t = \left[2\frac{t^5}{5} - \frac{2}{3}.\frac{t^7}{7}\right]_0^1 = \frac{32}{105}$	M1A1		
		$\frac{1}{2}\int y^2 \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = \int_0^1 \left(t^3 - \frac{2}{3}t^5 + \frac{1}{9}t^7\right)\mathrm{d}t = \left[\frac{t^4}{4} - \frac{t^6}{9} + \frac{t^8}{72}\right]_0^1 = \frac{11}{72}$	M1A1		
		Centroid is $\left(\frac{4}{7}, \frac{55}{192}\right)$ Or (0.571,0.286)	A1	6	**[12]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
10 (i)	Vertical asymptote. Oblique asymptote.	Asymptotes : $x = -1$ $y = px + 4 - p + (p-3)(x+1)^{-1} \Rightarrow y = px + 4 - p$	B1 M1A1	3	
(ii)	Obtains value of p. Sketches curve.	$p = 4 \Rightarrow$ x-axis is a tangent Correct location of turning points and asymptotes. Each branch.	B1 B1 B1B1	4	
(iii)	Proves required result. Sketches graph.	$p = 1 \Rightarrow y = x + 3 - 2(x+1)^{-1} \Rightarrow y' = 1 + 2(x+1)^{-2} \quad (\geqslant 1)$ Intersections on x-axis at $(-2 \pm \sqrt{3}, 0)$ Each branch.	M1A1 B1 B1B1	5	[12]
11E	Obtains all fifth roots.	$z = \cos\frac{2k\pi}{5} + \mathrm{i}\sin\frac{2k\pi}{5}, k = 0, \pm 1, \pm 2.$	B1B1	2	
	Simplifies expression.	$x^2 - 2\cos\frac{2\pi}{5}x + 1$	M1A1	2	
	Obtains factors.	$\left(x^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right)(x-1)$	M1A1	2	
	Solves quadratic in x^3.	$x^3 = \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} \pm \mathrm{i}\sin\frac{\pi}{3}$	M1A1 A1		
	Expresses them in polar form.	or $\cos\frac{7\pi}{3} \pm \mathrm{i}\sin\frac{7\pi}{3}$ or $\cos\frac{13\pi}{3} \pm \mathrm{i}\sin\frac{13\pi}{3}$	A1		
		$x = \cos\frac{\pi}{9} \pm \mathrm{i}\sin\frac{\pi}{9}, \cos\frac{7\pi}{9} \pm \mathrm{i}\sin\frac{7\pi}{9}, \cos\frac{13\pi}{9} \pm \mathrm{i}\sin\frac{13\pi}{9}$	M1A1	6	
	Finds factors.	$\left(x^2 - 2\cos\frac{\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{7\pi}{9} + 1\right)$ $\left(x^2 - 2\cos\frac{13\pi}{9} + 1\right)$ (ACF)	M1A1	2	[14]

Qu No	Commentary	Solution	Marks	Part Mark	Total
11O	Uses substitution	$v = y^3 \Rightarrow v' = 3y^2 \dfrac{dy}{dx} \Rightarrow v'' = 6y\left(\dfrac{dy}{dx}\right)^2 + 3y^2 \dfrac{d^2y}{dx^2}$	B1B1		
	to obtain v–x equation.	$\dfrac{1}{3}\dfrac{d^2v}{dx^2} - 2\dfrac{dv}{dx} + 3v = 25e^{-2x}$	M1		
		$\Rightarrow \dfrac{d^2v}{dx^2} - 6\dfrac{dv}{dx} + 9v = 75e^{-2x}$ (AG)	A1	4	
	Finds CF.	$m^2 - 6m + 9 = 0 \Rightarrow m = 3$	M1		
		$v = Ae^{3x} + Bxe^{3x}$	A1		
	Finds PI.	$v = ke^{-2x} \Rightarrow v' = -2ke^{-2x} \Rightarrow v'' = 4ke^{-2x}$	M1		
		$4k + 12k + 9k = 75k \Rightarrow k = 3$	A1		
		$v = Ae^{3x} + Bxe^{3x} + 3e^{-2x}$	A1		
	Uses initial conditions to find constants.	$x = 0,\ y = 2,\ v = 8 \Rightarrow 8 = A + 3 \Rightarrow A = 5$	B1		
		$v' = 15e^{3x} + 3Bxe^{3x} + Be^{3x} - 6e^{-2x}$	M1A1		
		$x = 0,\ y = 2,\ y' = 1 \Rightarrow v' = 12$			
		$12 = 15 + B - 6 \Rightarrow B = 3$	A1		
	Writes solution of y–x equation	$y^3 = v = 5e^{3x} + 3xe^{3x} + 3e^{-2x}$			
	explicitly.	$y = \left\{5e^{3x} + 3xe^{3x} + 3e^{-2x}\right\}^{\frac{1}{3}}$	A1	10	[14]