CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the October/November 2012 series

9231 FURTHER MATHEMATICS

9231/13 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2012 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Marks	Total
1	Use of:	$\sum_{n+1}^{2n} = \sum_{1}^{2n} - \sum_{1}^{n}$	M1		
	Use of:	$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$	M1		
	Obtains result.	$\frac{2n(2n+1)(4n+1)}{6} - \frac{n(n+1)(2n+1)}{6}$	A1		
		$= \frac{1}{6}n(2n+1)(8n+2-n-1) = \frac{1}{6}n(2n+1)(7n+1)$ (AG)	A1	4	[4]
2	Sets determinant $\neq 0$.	$\begin{vmatrix} a & 1 & 2 \\ 3 & -2 & 0 \\ 3 & -4 & -6a \end{vmatrix} \neq 0 \Rightarrow 12a^2 + 18a - 12 \neq 0$	M1A1		
	Factorises, or completes square.	$\Rightarrow 6(2a-1)(a+2) \neq 0$	M1		
	States result.	$a \neq \frac{1}{2}$ or -2 (Or by row operations.)	A1	4	[4]
3	Proposition.	H_N : $S_N = 1 - \frac{1}{(N+1)!}$			
	Proves base case.	$S_1 = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} \Rightarrow H_1$ is true.	B1		
	States inductive hypothesis.	H_k : Assume $S_k = 1 - \frac{1}{(k+1)!}$ is true.	B1		
	Proves inductive step.	$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)! - (k+2) + (k+1)}{(k+2)!}$	M1		
		$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+2)!}$ $\therefore H_k \Rightarrow H_{k+1}$.	A1		
	States conclusion.	$\therefore$ (By PMI H_n is) true for all positive integers N.	A1	5	[5]
4	Finds vector product.	$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	B1		
		$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$	M1A1	3	
	Finds area of triangle.	Area of triangle $ABC = \frac{1}{2}\left\lvert \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rvert = \frac{1}{2}\sqrt{3}$	M1A1		
	Finds length of perpendicular.	$\frac{1}{2}\sqrt{1^2+2^2+3^2}\,d = \frac{1}{2}\sqrt{3} \Rightarrow d = \sqrt{\frac{3}{14}}$	A1	3	[6]

Qu No	Commentary	Solution	Marks	Part Marks	To
5	Sketches graph.	Correct shape and orientation. Passing through (0,0) and (3,0)	B1 B1	2	
	Uses area of sector formula.	Area $= 2\times\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+2\cos\theta)^2\,\mathrm{d}\theta$	M1		
		$=\int_0^{\frac{\pi}{3}}(1+4\cos\theta+4\cos^2\theta)\mathrm{d}\theta$			
	Uses double angle formula Integrates.	$=\int_0^{\frac{\pi}{3}}(3+4\cos\theta+2\cos 2\theta)\,\mathrm{d}\theta$	M1		
		$=\left[3\theta+4\sin\theta+\sin 2\theta\right]_0^{\frac{\pi}{3}}$	A1		
	Obtains result.	$=\left[\pi+\frac{5}{2}\sqrt{3}\right]$	A1	4	[6]
6	Differentiates.	$\dot{x}=2t \qquad \dot{y}=t^3-\frac{1}{t}$	B1		
	Obtains $\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$	$(\dot{x})^2+(\dot{y})^2=4t^2+t^6-2t^2+\frac{1}{t^2}=\left(t^3+\frac{1}{t}\right)^2$	M1A1		
	Uses surface area formula about <u>y-axis</u>.	$S=\int 2\pi x\,\mathrm{d}s=2\pi\int_1^2 t^2\left(t^3+\frac{1}{t}\right)\mathrm{d}t=2\pi\int_1^2(t^5+t)\mathrm{d}t$	M1A1		
		$=2\pi\left[\frac{1}{6}t^6+\frac{1}{2}t^2\right]_1^2=2\pi\left\{\left[\frac{32}{3}+2\right]-\left[\frac{1}{6}+\frac{1}{2}\right]\right\}=24\pi$	M1A1	7	[7]
7		$(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=\alpha^2+\beta^2+\gamma^2\Rightarrow\sum\alpha\beta=1$	M1A1	2	
		Either Required equation is $x^3-4x^2+x+c=0$ $\Rightarrow\sum\alpha^3-4\sum a^2+4+3c=0$ $\Rightarrow 3c=56-34-4=18\Rightarrow c=6$ (AG)	 M1 M1 A1		
		Or $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$	 (M1)		
		(or some other appropriate identity, e.g. $(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$) $\Rightarrow\ldots\Rightarrow\alpha\beta\gamma=-6$	 (M1A1)		
		$\Rightarrow x^3-4x^2+x+6=0$ (AG)	A1		
		$\Rightarrow(x+1)(x-2)(x-3)=0\Rightarrow x=-1,2,3.$	M1A1	6	[8]

Qu No	Commentary	Solution	Marks	Part Marks	To
8	Re-write.	$1+z=2\cos^2\frac{1}{2}\theta+2\,\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$	M1A1		
	Obtains result.	$=2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+\mathrm{i}\sin\frac{1}{2}\theta\right)$ (AG)	A1	3	
	Use of Bin. Thm.	$(1+z)^n=1+\binom{n}{1}z+\binom{n}{2}z^2+\ldots+\binom{n}{n}z^n$	M1		
	Takes imaginary part and uses de M.	$\therefore \mathrm{Im}(1+z)^n=\binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\ldots+\binom{n}{n}\sin n\theta$	M1 M1A1		
	Applies initial result.	But $(1+z)^n=2^n\cos^n\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\frac{n}{2}\theta}$	B1		
	Equates imaginary parts to obtain result.	$\therefore \binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\ldots+\binom{n}{n}\sin n\theta$ $=2^n\cos^n\frac{1}{2}\theta\sin\frac{n}{2}\theta$	A1	6	[9]
9	States vertical asymptote.	Vertical asymptote is $x=2$.	B1		
	Divides and states oblique asymptote.	$\Rightarrow$ oblique asymptote is $y=x-1$.	M1 A1	3	
	Rearranges as quadratic in x.	$xy-2y=x^2-3x+3\Rightarrow x^2-(y+3)x+(3+2y)=0$	B1		
	Uses discriminant to obtain condition stated.	For real x, $B^2-4AC\geq 0$ $\therefore (y+3)^2-4(3+2y)\geq 0$ $\Rightarrow\ldots\Rightarrow(y-3)(y+1))\geq 0$ $\Rightarrow y\leq -1$ or $y\geq 3$ $\therefore$ no points for $-1<y<3$ (AG)	M1 A1 A1	4	
	Differentiates, puts = 0 and obtains x-values.	$y'=1-(x-2)^{-2}=0\Rightarrow x=1$ or 3 (Or uses $y=-1$ and 3 to obtain x-values.)	M1		
	States stationary points.	Stationary points are $(1,-1)$ and $(3,3)$	A1A1	3	
	Sketch.	One mark for each branch correctly placed.	B1B1	2	[12]

Qu No	Commentary	Solution	Marks	Part Marks	To
10	Differentiates implicitly.	$3x^2 + 3y^2 y' = 3y + 3xy'$	B1B1		
	Equates $\frac{\mathrm{d}y}{\mathrm{d}x}$ to zero.	$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x^2}{y^2 - 1} = 0$	M1		
	Obtains relationship.	$\Rightarrow y = x^2$	A1	4	
	Substitutes for y.	$\Rightarrow xy + y^3 = 3xy \Rightarrow y^3 = 2xy \Rightarrow y^2 = 2x \quad (y \neq 0)$	M1		
	Obtains x and y.	$\Rightarrow x^4 = 2x \Rightarrow x^3 = 2 \Rightarrow x = 2^{\frac{1}{3}}$ and $\Rightarrow y = 2^{\frac{2}{3}} \quad (x \neq 0)$	A1A1		
	Differentiates	$6x + 3y^2 y'' + 6y(y')^2 = 3y' + 3xy'' + 3y'$	B1 B1B1		
	Uses $y' = 0$.	$\Rightarrow 6x = y''(3x - 3y^2) \Rightarrow y'' = \frac{2x}{x(1 - x^3)}$	M1		
	Identifies maximum.	$\Rightarrow y'' = \frac{2}{1-2} = -2 \Rightarrow$ max	A1	8	
	(Other watertight methods for showing a maximum accepted.)				[12]

Qu No	Commentary	Solution	Marks	Part Marks	To
11	Verifies result.	$\frac{d}{dx}\left(-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+c\right)=-\frac{1}{3}\times\frac{3}{2}(1-x^2)^{\frac{1}{2}}\times(-2x)=x(1-x^2)^{\frac{1}{2}}$	B1	1	
	Correct parts.		M1		
	Integrates by parts.	$I_n=\int_0^1 x^n(1-x^2)^{\frac{1}{2}}\ dx\ =\ \int_0^1 x^{n-1}.x(1-x^2)^{\frac{1}{2}}\ dx$			
	Substitutes limits.	$=\left[-x^{n-1}.\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1+\int_0^1(n-1)x^{n-2}.\frac{1}{3}(1-x^2)^{\frac{3}{2}}\ dx$	M1A1		
	Obtains reduction formula.	$=\frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)(1-x^2)^{\frac{1}{2}}\ dx$	M1		
		$=\frac{n-1}{3}I_{n-2}-\frac{n-1}{3}I_n$			
		$\Rightarrow (n+2)I_n=(n-1)I_{n-2}$ (AG)	A1	5	
	Uses substitution correctly.	$x=\sin u\Rightarrow dx=\cos u\,du$ Limits: $x=0\Rightarrow u=0$ $\quad x=1\Rightarrow u=\frac{\pi}{2}$	M1		
	Uses double angle formula.	$\int_0^1(1-x^2)^{\frac{1}{2}}dx=\int_0^{\frac{\pi}{2}}\cos^2 u\,du$	A1		
	Integrates correctly.	$=\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos 2u+1)du$	M1		
	Uses reduction formula correctly.	$=\frac{1}{2}\left[\frac{\sin 2u}{2}+u\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}$ (AG)	M1A1	5	
		$\Rightarrow I_2=\frac{1}{4}\times\frac{\pi}{4}=\frac{\pi}{16}\Rightarrow I_4=\frac{1}{2}\times\frac{\pi}{16}=\frac{\pi}{32}$	M1A1	2	[13]

Qu No	Commentary	Solution	Marks	Part Marks	Total
12	**EITHER**				
	Use of these results.	$\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$ $\mathbf{ABe} = \mathbf{A}\mu\mathbf{e} = \mu\mathbf{Ae} = \mu\lambda\mathbf{e} = \lambda\mu\mathbf{e}$	M1 A1	2	
	Finds missing eigenvalues of **A**.	$\begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda = 0$	B1		
		$\begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \lambda = 1$	B1	2	
	Finds missing eigenvector of **A**.	$\lambda = 2 \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ -2 & -4 & -2 \end{vmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$	M1A1	2	
	Calculates eigenvectors of **B**	$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mu = 0$	B1		
		$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \mu = -3$	B1		
		$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \mu = -2$	B1		
	Uses initial result to find eigenvalues of **C**.(1 mark for one correct value, 2 marks for all three.)	$\therefore$ **C** has eigenvalues: $0 \times 0 = 0 \quad 1 \times (-3) = -3 \quad 2 \times (-2) = -4$	B2,1,0		
	Finds **P** and **D**.	$\mathbf{P} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ (OE) $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$	B1 M1A1	8	[14]

Qu No	Commentary	Solution	Marks	Part Marks	Total
12	**OR**				
	Finds complementary function.	$m^2+6m+13=0 \Rightarrow m=-3\pm 2\mathrm{i}$ $x=e^{-3t}(A\cos 2t+B\sin 2t)$	M1 A1		
	Finds particular integral.	$x=p\cos 2t+q\sin 2t$ $\dot{x}=-2p\sin 2t+2q\cos 2t$ $\ddot{x}=-4p\cos 2t-4q\sin 2t$	M1		
		$(9p+12q)\cos 2t+(9q-12p)\sin 2t=75\cos 2t$ $\Rightarrow p=3 \quad q=4$	M1A1 A1 A1	7	
	Finds general solution.	$x=e^{-3t}(A\cos 2t+B\sin 2t)+3\cos 2t+4\sin 2t$	B1		
	Uses initial conditions..	$x=5$ when $t=0 \Rightarrow 5=A+3 \Rightarrow A=2$ $\dot{x}=-3e^{-3t}(A\cos 2t+B\sin 2t)$ $+e^{-3t}(-2A\sin 2t+2B\cos 2t)-6\sin 2t+8\cos 2t$ $\dot{x}=0$ when $t=0 \Rightarrow 0=-6+8+2B \Rightarrow B=-1$	M1 A1 A1	4	
	Obtains solution.	$x=e^{-3t}(2\cos 2t-\sin 2t)+3\cos 2t+4\sin 2t$			
	Obtains limit.	As $t\to\infty$, $e^{-3t}\to 0$ $\therefore x\approx 3\cos 2t+4\sin 2t$ $\therefore x\approx 5\left(\frac{3}{5}\cos 2t+\frac{4}{5}\sin 2t\right)=5\cos\left(2t-\tan^{-1}\frac{4}{3}\right)$ (AG)	M1 M1A1	3	[14]